Advances in Engineering Software 99 (2016) 189–198

Contents lists available at ScienceDirect

Advances in Engineering Software

journal homepage: www.elsevier.com/locate/advengsoft

Explicit stress integration with streamlined drift reduction

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ARTICLE INFO

Article history:
Received 30 June 2015
Revised 30 March 2016
Accepted 16 May 2016

Keywords:
Explicit stress integration
Stress point method
Drift correction
Runge-kutta methods

ABSTRACT

The paper proposes an efficient method of drift correction in explicit stress integration schemes. The new method does not require significant changes to existing implementations and may, in fact, be regarded as a streamlining modification to standard drift correction schemes which may be used in tandem.

1. Introduction

In nonlinear finite element analysis, stresses are integrated from previously obtained strain increments. This process is commonly known as stress integration or the stress point method. Analytical stress integration is generally not possible for advanced elastoplastic constitutive models for soils and, therefore, approximate numerical stress integration is usually required. Stress integration schemes can be broadly classified into two broad categories: explicit and implicit. In implicit schemes, the gradients of the yield function and plastic potential are estimated at trial stress states and the stress-strain equation is treated as a set of nonlinear equations and solved by iteration. In explicit stress schemes, the gradients are estimated at known stress states and the stress-strain relations are treated as a set of differential equations and solved incrementally.

Implicit integration of constitutive relations for metals/soils has been discussed, among others, in [1–13]. Explicit integration techniques, on the other hand, have been described in [14–21]. Comparisons between the performance of implicit and explicit approaches are not common, though some data may be found e.g. in [22].

Explicit stress integration schemes usually do not integrate the stresses in one step. Instead, for reasons of accuracy, the strain increment is divided into subincrements (substeps) which are subsequently integrated consecutively [14–15,23]. The subincrementation process is generally linked with an error control mechanism where the error is estimated using numerical solutions of different orders of accuracy. Such explicit algorithms are referred to as substepping algorithms with adaptive error control, and they frequently use Runge-Kutta methods of different orders of accuracy.

At the end of the integration process, the computed stress state may not satisfy the yield function within a prescribed yield tolerance (*FTOL*) due to approximation errors, a problem which is known as yield surface drift. The tolerance *FTOL* may be defined as either a small positive constant or as a dimensionless fraction of the normalised size of the yield locus [20,24].

A simple method to reduce the drift (i.e. reduce the non-zero value of the yield function after integration) is to increase the accuracy of the explicit stress integration scheme. This option, however, may be computationally expensive, especially when excessive drift occurs only occasionally. In fact, when drift correction is required in most subincrements, it is likely that the value of *FTOL* is too stringent for the error tolerance used to integrate the stresses. When drift correction is required only occasionally, the most efficient way to restore the stresses to the yield surface is via a special drift correction algorithm. In such cases, the correction may be applied at the end of each integration step or at the end of each integration substep, and is a separate part of the stress integration process (see, for example, [25]).

Vrh et al. [26], based on the idea of Halilovič et al. [27] proposed a NICE method to reduce the yield surface drift during the integration of classical elastoplastic models. This method has been used with some success, e.g. in [28–31]. Furthermore, the method of Vrh et al. [26] has been extended by Halilovic et al. [32] to improve the accuracy of the integration. This paper shows a similar

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http://dx.doi.org/10.1016/j.advengsoft.2016.05.011

development as it extends the idea of Vrh et al. [26] to include higher-order explicit stress integration of elasto-plastic models. However, here the NICE method is used to improve the integration schemes that are based on Runge-Kutta methods, which have been widely used for advanced constitutive models. A key feature of the proposed approach is that the drift correction is incorporated in the stress integration algorithm in such a way that few additional calculations are required. The new algorithm enhances the performance of the Runge-Kutta schemes. In addition, it preserves typical features of explicit stress integration with Runge-Kutta methods such as automatic error control (not available in [32]), the possibility of using high-order schemes easily (without the need for computing derivatives, unlike [32]), and access to the highly optimised Runge-Kutta pairs that are widely used in a great many areas.

2. Description of the NICE drift correction scheme for explicit stress integration

The drift correction algorithm proposed in [25] is well suited for complex elasto-plastic constitutive models, such as those that are used to model the response of geomaterials. This scheme is used often in explicit stress integration with automatic substepping and error control, where the integration algorithm is based on a Runge-Kutta method. In such cases, the stress is integrated in series of substeps, with each substep corresponding to a strain subincrement. At the end of each substep, the stress state typically does not satisfy the yield locus condition $F=0$ exactly. Once this deviation is deemed to be too large, a drift correction algorithm is called and the stress state is restored to the yield surface to within a prescribed tolerance. This approach requires additional calculations and, in some highly nonlinear cases, the time spent on drift correction may be substantial.

The proposed approach implements the drift correction without any additional computations. Moreover, it will be shown that this new procedure is roughly equivalent to a version of the Potts & Gens given in [25] where the drift correction is performed during integration of the subsequent stress (sub)increment. The proposed algorithm, presented in Section 3, relies on the NICE approach [26] which extends the usual consistency condition

$$dF(\boldsymbol{\sigma},\,\mathbf{h})=0 \tag{1}$$

to one where the initial value of yield function is taken into account

$$F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)+dF(\boldsymbol{\sigma},\,\mathbf{h})=F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)+\left(\frac{\partial F(\boldsymbol{\sigma},\,\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T d\boldsymbol{\sigma} + \left(\frac{\partial F(\boldsymbol{\sigma},\,\mathbf{h})}{\partial\mathbf{h}}\right)^T d\mathbf{h}=0 \tag{2}$$

Here it is assumed that the yield locus depends on the stresses $\boldsymbol{\sigma}$ and a set of hardening parameters $\mathbf{h}$, with the subscript 0 indicating values of the stresses and hardening parameters at the beginning of the (sub)increment.

This approach leads to some modifications in the standard integration procedure as the elasto-plastic matrix is based on the new consistency condition (2). Following conventional practice, the stresses are computed as

$$d\boldsymbol{\sigma}=\mathbf{D}^e d\boldsymbol{\varepsilon}^e=\mathbf{D}^e(d\boldsymbol{\varepsilon}-d\boldsymbol{\varepsilon}^p) \tag{3}$$

where $\mathbf{D}^e$ is the elastic stress-strain matrix and the superscripts e and p denote elastic and plastic respectively. The plastic strains are given by the standard flow rule

$$d\boldsymbol{\varepsilon}^p=\lambda\frac{\partial Q}{d\boldsymbol{\sigma}} \tag{4}$$

where λ is a scalar plastic multiplier and Q is the plastic potential, while the hardening parameter increment is dependent on the plastic strain increment according to

$$d\mathbf{h}=\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}d\boldsymbol{\varepsilon}^p \tag{5}$$

Introducing Eqs. (3)–(5) into (2) results in the following expressions for the plastic multiplier and elasto-plastic tangent matrix:

$$\lambda=\frac{F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)}{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e\frac{\partial Q}{d\boldsymbol{\sigma}}-\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\mathbf{h}}\right)^T\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}} + \frac{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e d\boldsymbol{\varepsilon}}{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e\frac{\partial Q}{d\boldsymbol{\sigma}}-\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\mathbf{h}}\right)^T\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}} \tag{6}$$

$$d\boldsymbol{\sigma}=\mathbf{D}^e d\boldsymbol{\varepsilon}^e=\mathbf{D}^e\left(\boldsymbol{\varepsilon}-\lambda\frac{\partial Q}{d\boldsymbol{\sigma}}\right)=\mathbf{D}^{ep}d\boldsymbol{\varepsilon}-\mathbf{D}^e d\lambda\frac{\partial Q}{d\boldsymbol{\sigma}} = \mathbf{D}^{ep}d\boldsymbol{\varepsilon}-\frac{\mathbf{D}^e F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)\frac{\partial Q}{d\boldsymbol{\sigma}}}{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e\frac{\partial Q}{d\boldsymbol{\sigma}}-\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\mathbf{h}}\right)^T\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}} \tag{7}$$

Therefore, the final stresses depend on the same elasto-plastic matrix as before, but with an additional correction that depends on the amount of drift. This correction

$$d\boldsymbol{\sigma}_{NICE}=-\mathbf{D}^e d\lambda\frac{\partial Q}{d\boldsymbol{\sigma}}=-\frac{\mathbf{D}^e F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)\frac{\partial Q}{d\boldsymbol{\sigma}}}{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e\frac{\partial Q}{d\boldsymbol{\sigma}}-\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\mathbf{h}}\right)^T\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}} \tag{8}$$

may be applied at the end of each step, thus reducing the drift occurring during the integration. When an elasto-plastic constitutive model with hardening is considered, the adjustment in the scalar plastic multiplier leads also to a correction in the hardening parameter

$$d\mathbf{h}_{NICE}=\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\mathrm{d}\boldsymbol{\varepsilon}=\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}\delta\lambda = \frac{\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}F(\boldsymbol{\sigma}_0,\,\mathbf{h}_0)}{\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\boldsymbol{\sigma}}\right)^T\mathbf{D}^e\frac{\partial Q}{d\boldsymbol{\sigma}}-\left(\frac{\partial F(\boldsymbol{\sigma},\mathbf{h})}{\partial\mathbf{h}}\right)^T\frac{\partial\mathbf{h}}{\partial\boldsymbol{\varepsilon}^p}\frac{\partial Q}{d\boldsymbol{\sigma}}} \tag{9}$$

Both of the above corrections arise because of the modified consistency condition and can be computed during the calculations very cheaply (as all the terms are needed for the standard integration anyway). It is interesting to note that this scheme, if applied separately and iteratively before the next substep calculation, like a standard correction procedure, is identical to the "consistent" correction procedure described by Potts & Gens [25]; provided all the derivatives are evaluated at the updated stresses (i.e. the final stresses for the last accepted integration subincrement).

3. NICE drift correction scheme for Runge-Kutta methods

In explicit stress integration, the strain increment $\Delta\boldsymbol{\varepsilon}$ is divided into subincrements $\delta\boldsymbol{\varepsilon}_i$, such that the error in the corresponding stress increment $\delta\boldsymbol{\sigma}_i$ is below a user specified value. In each of the subincrements $\delta\boldsymbol{\varepsilon}_i$, the stresses are integrated with a Runge-Kutta scheme in several stages. The simplest second order Runge-Kutta method has just two stages, with the number of stages NoS increasing with increasing order of accuracy. In general, the integration process for each strain subincrement can be summarised as [20]

$$\delta\boldsymbol{\sigma}^{(j)}=\mathbf{D}^{ep(j)}\left(\boldsymbol{\varepsilon}_0+c^{(j)}\delta\boldsymbol{\varepsilon}_i,\ \boldsymbol{\sigma}_0+\sum_{k=1}^{j-1}a^{(jk)}\delta\boldsymbol{\sigma}^{(k)},\ \mathbf{h}_0+\sum_{k=1}^{j-1}a^{(jk)}\delta\mathbf{h}^{(k)}\right)\delta\boldsymbol{\varepsilon}$$