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Integrating rotation and angular velocity from curvature

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ABSTRACT

The problem of integrating the rotational vector from a given angular velocity vector is met in such diverse fields as the navigation, robotics, computer graphics, optical tracking and non-linear dynamics of flexible beams. For example, if the numerical formulation of non-linear dynamics of flexible beams is based on the interpolation of curvature, one needs to derive the rotation from the assumed curvature field. The relation between the angular velocity and the rotation is described by the first-order quasi-linear differential equation. If the rotation is given, the related angular velocity is obtained by the differentiation. By contrast, if the angular velocity is given, the related rotations are obtained by the integration. The exact closed-form solution for the rotation is only possible if the angular velocity is constant in time. In dynamics of non-linear flexible spatial beams, the problem of integrating rotations from a given angular velocity becomes even more complex because both the angular velocity and the curvature need simultaneously be integrated and are both functions of space and time. As the angular velocity and the curvature are assumed to be analytic functions, they must satisfy certain integrability conditions to assure the unique rotation is obtained from the two differential equations. The objective of the present paper is to derive approximate, yet closed-form solutions of the following problem: for a given curvature vector, determine both the rotation and the angular velocity. In order to avoid the singularity of kinematic relations, the quaternions are used for the parametrization of rotations, and the integrations are partly performed in the four-dimensional quaternion space. The resulting closed-form expressions for the rotational and angular velocity quaternions are ready to be used in the finite-element formulations of the dynamics of flexible spatial beams as interpolating functions. The present novel solution is assessed by comparisons of the numerical results with analytical solutions for variety of oscillating curvature functions, as well as with the solutions of the quaternion-based midpoint integrator and the Runge–Kutta-based Crouch–Grossman geometrical methods CG3 and CG4.

1. Introduction

The problem of integrating the rotational vector from a given time-dependent angular velocity vector is met in such diverse fields as the navigation, robotics, computer graphics, and dynamics of rigid bodies. For example, in a strapdown inertial navigation system [1–5], the body-fixed gyroscopes measure the instantaneous angular velocity of a space vehicle, which is on-line numerically integrated to obtain the instantaneous coordinate transformation matrix between the body-fixed and the spatial coordinate systems needed to establish the attitude of the vehicle. Obviously, having an accurate, stable and computationally efficient integration is vital in controlling the spatial position of such systems in flight. The same integration problem must be solved in real-time optical tracking of a human body motion, which is related to problems in robotics and computer graphics [6,7].

A further significant example is met in computational structural mechanics in the study of non-linear dynamics of flexible beams if the numerical formulation is based on the interpolation of curvature as in, e.g. [8,9]. There one needs to derive the rotation from the assumed curvature field when given in terms of the arc-length parameter. This is essentially the same problem as described above, because the differential equations for both the angular velocity and the curvature in terms of the rotation are formally the same.

The relation between the angular velocity (or curvature) and the rotation is described by the first-order quasi-linear differential equation. When the rotation is a given function, the related angular velocity (or curvature) is obtained by the differentiation. When the angular velocity (or curvature) is given, the related rotations must be obtained by the integration. The exact closed-form solution for the rotation is possible for angular velocities and curvatures,

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respectively, only if some further conditions are satisfied; e.g. if the angular velocity is constant in time. An alternative solution in terms of an infinite series is known for any analytic angular velocity (or curvature), see, e.g. [1,10]. Such a solution is computationally inefficient. Various numerical time-integration schemes have been introduced and their performance discussed, see, e.g. [11,12] or [13,10] for the references related to the structural mechanics problems. In [14] it is investigated how the choice of the parametrization of rotations effects accuracy and computational efficiency of various numerical time-integration schemes by comparing three different rotation parameterizations, i.e. the rotational vector [15,16,8], the Argyris tangential vector [15] and the rotational quaternion [17], if combined with two alternative mid-point rules [13,14] or the classical fifth-order Runge–Kutta method. Andrle and Crassidis [11] employ the Crouch–Grossman geometric methods and compare their results with the Runge–Kutta algorithms. Chiou et al. [12] developed a family of general high-order numerical time integrators that exactly preserve the constraint of the rotational quaternion.

In dynamics of non-linear flexible spatial beams, the problem of integrating rotations from a given angular velocity becomes even more complex because both the angular velocity and the curvature need simultaneously be integrated and both are functions of independent variables x and t. As the angular velocity and the curvature are assumed to be analytic functions, they must satisfy certain integrability conditions to assure that the unique rotation is obtained from the two differential equations. Our objective here is thus to derive approximate yet closed-form solutions of the following problem: for a given curvature vector dependent on two variables, x and t, determine both the rotation and the angular velocity. In order to avoid the singularity of the kinematic relations [15,18], the quaternions are used for the parametrization of rotations in the present paper, and the integrations are partly performed in the four-dimensional quaternion space. The resulting closed-form expressions for the rotational and angular velocity quaternions could be used in refined, finite-element type of numerical formulations of the dynamics of flexible spatial beams as the interpolating functions.

2. Quaternions

Quaternions have recently often been discussed in structural mechanics literature, see, e.g. [19,20,17], so that only equations essential for the present study are given. A systematic presentation of quaternions can be found elsewhere [21,22].

The quaternion, $\widehat{a}$, is a four-component element, defined as a formal sum of a scalar and a vector: $\widehat{a} = a_0 + \vec{a}$. For two arbitrary quaternions, $\widehat{a} = a_0 + \vec{a}$ and $\widehat{b} = b_0 + \vec{b}$, and a scalar, λ, the following operations are defined:

(i) sum: $\widehat{a} + \widehat{b} := (a_0 + b_0) + \left(\vec{a} + \vec{b}\right) = \widehat{b} + \widehat{a}$,

(ii) multiplication by a scalar: $\lambda\widehat{a} := \lambda a_0 + \lambda\,\vec{a}$,

(iii) multiplication by a quaternion: $\widehat{a} \circ \widehat{b} := \left(a_0 b_0 - \vec{a}\cdot\vec{b}\right) + \left(b_0\,\vec{a} + a_0\,\vec{b} + \vec{a}\times\vec{b}\right)$.

The quaternion multiplication is associative: $\widehat{a} \circ (\widehat{b} \circ \widehat{c}) = (\widehat{a} \circ \widehat{b}) \circ \widehat{c}$, yet it is not commutative, because $\widehat{a} \circ \widehat{b} - \widehat{b} \circ \widehat{a} = 2\,\vec{a}\times\vec{b}$. Quaternions are elements of the 4D vector space.

The *null* quaternion is defined as $\widehat{0} = 0 + \vec{0}$. The *identity* quaternion is $\widehat{1} = 1 + \vec{0}$. Hence $\widehat{a} + \widehat{0} = \widehat{0} + \widehat{a} = \widehat{a}$, $\widehat{0} \circ \widehat{a} = \widehat{a} \circ \widehat{0} = \widehat{0}$ and $\widehat{1} \circ \widehat{a} = \widehat{a} \circ \widehat{1} = \widehat{a}$. The *conjugated* quaternion is defined as $\widehat{a}^* = a_0 - \vec{a}$. This implies that $\left(\widehat{a} \circ \widehat{b}\right)^* = \widehat{b}^* \circ \widehat{a}^*$. The *norm* of a quaternion is defined as $\|\widehat{a}\| = \sqrt{\widehat{a} \circ \widehat{a}^*} = \sqrt{a_0^2 + \left\|\vec{a}\right\|^2}$. The quaternion whose norm is 1 is the *unit* quaternion. In what follows, the unit quaternion will play a remarkable role and will be denoted by $\widehat{q}$. The *inverse* of a quaternion, $\widehat{a}^{-1}$, satisfies the condition $\widehat{a}^{-1} \circ \widehat{a} = \widehat{a} \circ \widehat{a}^{-1} = \widehat{1}$; hence [22]

$$\widehat{a}^{-1} = \frac{\widehat{a}^*}{\|\widehat{a}\|^2}. \tag{1}$$

If $\|\widehat{a}\| = 1$,

$$\widehat{a}^{-1} = \widehat{a}^* \quad \text{and} \quad \widehat{a} \circ \widehat{a}^* = \widehat{a}^* \circ \widehat{a} = \widehat{1}. \tag{2}$$

The scalar part of a *pure* quaternion is zero: $\widehat{a}_{\mathrm{pure}} = 0 + \vec{a}$. This implies $\widehat{a}^*_{\mathrm{pure}} = -\widehat{a}$.

A quaternion can also be written in an alternative *polar form* $\widehat{a} = \|\widehat{a}\|(\cos\theta + \vec{a}_n \sin\theta)$, where $\vec{a}_n = \frac{\vec{a}}{\|\vec{a}\|}$ is the unit vector, $\vec{n}$; angle θ is extracted from the quaternion using $\cos\theta = \frac{a_0}{\|\widehat{a}\|}$ and $\sin\theta = \frac{\|\vec{a}\|}{\|\widehat{a}\|}$. When the quaternion is unitary, its norm is 1; thus

$$\text{if} \quad \|\widehat{q}\| = 1, \quad \text{then} \quad \widehat{q} = \cos\theta + \vec{n}\,\sin\theta. \tag{3}$$

Now we show that the unit quaternion $\widehat{q} = q_0 + \vec{q}$ represents the rotation of a quaternion in 4D. Because the quaternion product is not commutative, the left and the right multiplications of a quaternion, $\widehat{a}$, with the unit quaternion, $\widehat{q}$, yield two different quaternions

$$\widehat{a}_{\mathrm{L}} = \widehat{q} \circ \widehat{a}, \qquad \widehat{a}_{\mathrm{R}} = \widehat{a} \circ \widehat{q}, \tag{4}$$

but their norms remain equal to $\|\widehat{a}\|$:

$$\|\widehat{a}_{\mathrm{L}}\|^2 = \widehat{a}_{\mathrm{L}} \circ \widehat{a}^*_{\mathrm{L}} = (\widehat{q} \circ \widehat{a}) \circ (\widehat{q} \circ \widehat{a})^* = \widehat{q} \circ \widehat{a} \circ \widehat{a}^* \circ \widehat{q}^* = \|\widehat{a}\|^2\,\widehat{q} \circ \widehat{q}^* = \|\widehat{a}\|^2,$$
$$\|\widehat{a}_{\mathrm{R}}\|^2 = \widehat{a}_{\mathrm{R}} \circ \widehat{a}^*_{\mathrm{R}} = (\widehat{a} \circ \widehat{q}) \circ (\widehat{a} \circ \widehat{q})^* = \widehat{a} \circ \widehat{q} \circ \widehat{q}^* \circ \widehat{a}^* = \|\widehat{a}\|^2.$$

This shows that both the left and the right multiplication with the unit quaternion represent the rotation of a quaternion in 4D. The resulting quaternions are not pure quaternions, however.

Similarly, the 3D rotation of a vector must result in a 3D vector whose length is preserved. Consequently, if a vector $\vec{a}$ is represented by a pure quaternion, $\widehat{a} = 0 + \vec{a}$, it should remain such in the rotated position. Hence, the 4D rotation of a pure quaternion should result in a pure quaternion. By performing explicit multiplications it can be shown that the following composition of two subsequent multiplications with the unit quaternion $\widehat{q}$

$$\widehat{a}_{\mathrm{rot}} = \widehat{q} \circ \widehat{a} \circ \widehat{q}^* \tag{5}$$

yields the pure quaternion $\widehat{a}_{\mathrm{rot}} = 0 + \vec{a}_{\mathrm{rot}}$. As, furthermore, the norm of $\widehat{a}$ does not change in the 4D rotation (5)

$$\begin{aligned}\|\widehat{a}_{\mathrm{rot}}\|^2 &= \widehat{a}_{\mathrm{rot}} \circ \widehat{a}^*_{\mathrm{rot}} = \widehat{q} \circ \widehat{a} \circ \widehat{q}^* \circ (\widehat{q} \circ \widehat{a} \circ \widehat{q}^*)^* \\ &= \widehat{q} \circ \widehat{a} \circ \widehat{q}^* \circ \widehat{q} \circ \widehat{a}^* \circ \widehat{q}^* = \|\widehat{a}\|^2,\end{aligned}$$

$\widehat{q} \circ \widehat{a} \circ \widehat{q}^*$ represents the rotation of vector $\vec{a}$ in 3D indeed.

Because $\widehat{q}$ and $\widehat{q}^*$ each rotates by the same angle [22], their composition (5) results in the double angle rotation. Thus the double left–right operation (5) with the unit quaternion

$$\widehat{q}(\vartheta, \vec{n}) = \cos\frac{\vartheta}{2} + \vec{n}\,\sin\frac{\vartheta}{2} \tag{6}$$

on an arbitrary pure quaternion $\widehat{a} = 0 + \vec{a}$ results in a pure quaternion $\widehat{a}_{\mathrm{rot}} = \widehat{q} \circ \widehat{a} \circ \widehat{q}^* = 0 + \vec{a}_{\mathrm{rot}}$, being rotated from $\widehat{a}$ by angle ϑ about axis $\vec{n}$.