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Asynchronous grid computation for American options derivatives

M. Chau [a], R. Couturier [b], J. Bahi [b], P. Spiteri [c,*]

[a] Advanced Solutions Accelerator, 199 rue de l'Oppidum, 34170 Castlenau le Lez, France
[b] FEMTO-ST INSTITUTE, IUT Belfort-Montbéliard, Rue Engel Gros, BP 527, 90016 Belfort cedex, France
[c] ENSEEIHT-IRIT, 2 rue Charles Camichel, 31071 Toulouse Cedex, France

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ABSTRACT

In relation with the mathematics of financial applications, the present study deals with the solution of the time dependent obstacle problem defined in a three-dimensional domain; this problem arises in the pricing of American options derivatives. In order to solve very quickly large scale algebraic systems derived from the discretization of the obstacle problem, the parallelization of the numerical algorithm is necessary. So, we present parallel synchronous, and more generally asynchronous, iterative algorithms to solve this problem. For the considered problem, arguments implying the convergence of parallel synchronous and asynchronous algorithms are given in a general framework. Finally, computational experiments on GRID'5000, the French national grid, are presented and analyzed. They allow us to compare both synchronous and asynchronous versions with local and distributed clusters and to show the interest of such methods in the context of grid computing.

1. Introduction

The goal of the present study is to solve the discretized time dependent obstacle problem related to financial applications, more specifically the pricing of American options derivatives (see [1]). Note also that the same obstacle problem is likely to occur in many other applications such as mechanics and free boundary problems. Sequential algorithms have been studied for example in [2].

Taking into account the large scale of the systems to solve, we concentrate here on iterative parallel synchronous and asynchronous algorithms. These methods are used in order to reduce the computation elapsed time. For such parallel iterative methods the convergence is ensured when the operators arising in the mathematical problem are discretized by appropriate schemes. More precisely for the discretization of the evolution part of the mathematical problem, an implicit time marching scheme is carried out; then, we have to solve a sequence of stationary obstacle problems, each stationary obstacle problem corresponding to the solution at each time step. Furthermore well adapted schemes are used in order to achieve the spatial discretization of the remaining operators present in the mathematical model, i.e. the spatial part of the mathematical problem. Then, it can be shown that in the systems to solve, at each time step, we have to solve large linear algebraic systems in which the matrix to invert is an M-matrix [3]. Then, the convergence of parallel asynchronous and synchronous iterative fixed point methods applied to the solution of the considered problem can be proven by applying our general results; more precisely the convergence of the considered algorithm is studied either by contraction techniques [4–6] or by partial ordering techniques [7–9], in a theoretical framework well adapted to distributed computation. Note that, practically, it appears that asynchronous algorithms, compared to the synchronous ones, reduce idle times due to less synchronizations between the processors. Moreover, from an algorithmic point of view, two kinds of parallel asynchronous and synchronous methods are implemented: the projected Richardson's method and the projected block relaxation method.

Implementation of the considered algorithms is carried out on a distributed memory multiprocessor. Communications are managed with M.P.I. More specifically, computational experiments are performed on GRID'5000, the French national grid (see [10]). Asynchronous and synchronous versions of the parallel algorithms are compared; their efficiency is analyzed. In the present study, we have mainly considered parallel experiments carried out on distant and heterogeneous clusters. In such a context, the classical notions of speed-up and efficiency are not relevant. So we have used the ratio between the elapsed times obtained with synchronous and asynchronous algorithms in order to compare their respective behaviors. Moreover, for the analysis of the considered iterative algorithms, we have also considered the impact of the communication on the elapsed times.

Note also that due to the considered parallel algorithms, we consider a subdomain method with no overlapping between

* Corresponding author. Tel.: +33 0534322166; fax: +33 0534322157.
E-mail addresses: mchau@asa-sas.com (M. Chau), raphael.couturier@univ-fcomte.fr (R. Couturier), jacques.bahi@univ-fcomte.fr (J. Bahi), Pierre.Spiteri@enseeiht.fr (P. Spiteri).

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the subdomains. According to [11–13] we can also consider, for the solution of the obstacle problem, the Schwarz alternating method with overlapping between the subdomains; note that these works do not take into account asynchronous communications.

The paper is organized as follows: in Section 2 the model problem is presented and several equivalent formulations of the same problem are given. The next section is devoted to the description of the parallel synchronous and more generally asynchronous iterative algorithms; in particular the asynchronous projected Richardson's method and the projected asynchronous block relaxation method are presented and arguments which ensure their convergence are briefly given. Finally, in Section 4, experimental results on grid environment are presented and analyzed.

2. The model problem

In order to illustrate the present financial study, we consider here the case of some American options modeled by the Black–Scholes equations [1]. The classical Black–Scholes equation is a boundary value problem describing the evolution of call or put options in the field of mathematics of financial contracts. Among the many descriptions of financial option contracts, we concentrate here on American options which may be exercised at any time prior to expiry, i.e. when the time τ takes any value between 0 and T, where T denotes the expiry date. Classically, an American option is modeled by the following retrograde time dependent nonlinear convection–diffusion equation:

$$\begin{cases} \frac{\partial v}{\partial \tau} + \left(r - \frac{\sigma^2}{2}\right)\nabla v + \frac{\sigma^2}{2}\Delta v - rv \geqslant 0, \quad v \geqslant \phi, e.w.\ in\ [0,T]\times\mathbb{R}^n \\ \left(\frac{\partial v}{\partial \tau} + \left(r - \frac{\sigma^2}{2}\right)\nabla v + \frac{\sigma^2}{2}\Delta v - rv\right)(v-\phi) = 0, \quad e.w.\ in\ [0,T]\times\mathbb{R}^n \\ v(T,S) = \phi \end{cases} \tag{1}$$

where, e.w. means every where, $\phi = \phi(S) = \max\,(S - K, 0)$ in the case of call option or $\phi = \max\,(K - S, 0)$ in the case of put option; in the previous equations v denotes the value of the considered option, i.e. a call or a put option; $v = v(\tau, S)$ is a function of the current value of the underlying asset S and of the time τ. Note also that the considered option also depends on the following parameters:

- r the interest rate,
- σ the volatility of the underlying asset, σ being in fact the instantaneous standard deviation of the price with respect to the exercise price K, classically called strike and fixed beforehand; in fact σ characterizes the uncertainty of the option's behavior.

Note that the previous boundary value problem is not defined on a bounded domain, but is defined on the unbounded domain $\mathbb{R}^n, n \geqslant 1$. This difficulty is solved by considering the problem defined on a bounded domain $\Omega \subset \mathbb{R}^n$, and it can be proven that the solution of the retrograde time dependent convection–diffusion equation defined on the bounded domain Ω converges to the solution of problem (1) when the measure of Ω tends to infinity (see [14]).

Another particularity of the problem to solve, is that the value of the option is not known at the initial time $\tau = 0$; only the value $v(T, S)$ is known. In fact the problem consists in computing $v(0, S)$.

These previous two issues can be resolved, firstly by considering problem (1) defined in a bounded large domain Ω and secondly by a change of variables concerning the time, which consists in replacing the variable τ by a variable $t = T - \tau$. Thus, problem (1) is replaced by a classical time dependent convection–diffusion problem modeled as follows:

$$\begin{cases} \frac{\partial v}{\partial t} - \left(r - \frac{\sigma^2}{2}\right)\nabla v - \frac{\sigma^2}{2}\Delta v + rv \geqslant 0, v \geqslant \phi, e.w.\ in\ [0,T]\times\Omega \\ \left(\frac{\partial v}{\partial t} - \left(r - \frac{\sigma^2}{2}\right)\nabla v - \frac{\sigma^2}{2}\Delta v + rv\right)(v-\phi) = 0, e.w.\ in\ [0,T]\times\Omega \\ v(0,S) = \phi \\ \text{B.C. on } v(t,S) \text{ defined on } \partial\Omega \end{cases} \tag{2}$$

where B.C. describes the boundary conditions on the boundary $\partial\Omega$ of the domain Ω. Practically, the Dirichlet condition (where v is fixed on $\partial\Omega$) or the Neumann condition (where the normal derivative of v is fixed on $\partial\Omega$) are classically considered.

Remark 1. In the previous problem (2) note that the convection–diffusion operator is not self adjoint. Nevertheless, since the coefficients arising in the operator are constant, then by a classical change of variables, we can formulate the same problem by the way of a self adjoint operator. Indeed, consider the following time dependent convection–diffusion operator:

$$\frac{\partial v}{\partial t} + b^t\nabla v - \nu\Delta v + cv = g, e.w.\ in\ [0,T]\times\Omega, \quad c \geqslant 0,$$

where $b = \{b_1, b_2, b_3\}$ and consider also the following general change of variables $v = e^{\beta t}.e^{a}.u$, where a is defined by $a = \frac{b^t S}{2\nu}$; then, the previous time dependent convection–diffusion operator is changed as follows

$$\frac{\partial u}{\partial t} - \nu\Delta u + \left(\frac{\|b\|_2^2}{4\nu} + c + \beta\right)u = e^{-\beta t}.e^{-a}.g = f,$$

where $\|b\|_2$ denotes the euclidean norm; then, by using this change of variables, the time dependent convection–diffusion operator is changed in a time dependent diffusion operator, which has the major property of being a self adjoint operator. We can also choose β by various ways; for example if β is chosen positive, then we can take any value for β, for example $\beta = 1$; if β is chosen negative we can transform the operator as follows

$$\frac{\partial u}{\partial t} - \nu\Delta u = e^{-\beta t}.e^{-a}.g = f,$$

by taking $\beta = -\left(\frac{\|b\|_2^2}{4\nu} + c\right)$. Note also that, in the same way, if we consider the stationary convection–diffusion operator

$$b^t\nabla v - \nu\Delta v + cv = g, e.w.\ in\ [0,T]\times\Omega, \quad c \geqslant 0;$$

then the change of variables $v = e^{a}.u$, where a is defined by $a = \frac{b^t S}{2\nu}$, leads to the following expression of this operator

$$-\nu\Delta u + \left(\frac{\|b\|_2^2}{4\nu} + c\right)u = e^{-a}.g = f,$$

corresponding to a self adjoint operator.

Now, the application of the previous change of variables to the studied obstacle problem, leads to the following formulation of the American option problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\Delta u + \left(r + \frac{3}{2\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2\right)u \geqslant 0, u \geqslant \bar{\phi}, e.w.\ in\ [0,T]\times\Omega \\ \left(\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\Delta u + \left(r + \frac{3}{2\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2\right)u\right)(u - \bar{\phi}) = 0, e.w.\ in\ [0,T]\times\Omega \\ u(0,S) = \bar{\phi} \\ u(t,S) = 0\ e.w.\ on\ \partial\Omega \end{cases} \tag{3}$$

when we consider the homogeneous Dirichlet boundary condition and