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Brief paper

Collaborative scalar-gain estimators for potentially unstable social dynamics with limited communication☆

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ABSTRACT

In this paper, we study the estimation of potentially unstable social dynamics—e.g., social and political movements, environmental and health hazards, and global brands; when they are observed by a geographically distributed set of agents. We are interested in scenarios when the information exchange among the agents is limited. This paper considers a generalization of distributed estimation to *vector* (non-scalar) and *dynamic* (non-static) cases. As we will show, *when the state-vector evolves over time, the information flow over the communication network may not be fast enough to track this evolution*. In this context, the key questions we address are: (i) can a distributed estimator with limited communication track an unstable system? and; (ii) what is the cutoff point beyond which the given observations and the agent topology may not result into a bounded estimation error? To address these questions, we present a scalar-gain estimator and characterize the relation between the system instability and communication/observation infrastructure. We derive and analyze the aforementioned cutoff point as the Scalar Tracking Capacity, and further show that *unstable vector systems can be distributedly estimated with bounded error*.

1. Introduction

Social networks have recently seen a tremendous activity in the control and signal processing communities. A lot of attention has been devoted to the modeling and learning of relevant social phenomena, e.g., opinions, fashion, and rumors; while market trends in stocks and trading have also been studied, see Friedkin (1998), Newman, Barabasi, and Watts (2006), Urry (2002) and references therein. Typically, it is assumed that the social network consists of agents that make partial observations of the underlying phenomena while communicating over a sparse graph.

Several social modeling and interaction strategies have been formulated. Static phenomenon that do not evolve over time can be found in DeGroot (1974), Jadbabaie, Moolavi, Sandroni, and Tahbaz-Salehi (2012), Ram, Veeravalli, and Nedic (2010), Kar, Moura, and Ramanan (2012), among others. For the estimation of dynamic social models, related work includes Kalman-consensus filters (Kirti & Scaglione, 2008; Olfati-Saber, 2005, 2007), and learning in social networks (Acemoglu, Nedic, & Ozdaglar, 2008; Acemoglu & Ozdaglar, 2011). However, the former is restricted to a large number of communication iterations (in order to reach average-consensus Xiao & Boyd, 2004) between every two steps of the dynamics, see Fig. 1-left, while the latter assumes a neutrally-stable scalar system. However, many multi-agent systems and social interactions are observed to have unstable dynamics such as social and political movements, environmental and health hazards, and global brands (Tyukin, Prokhorov, & van Leeuwen, 2007; Urry, 2002).

Given the existing development, we address the natural transition to the learning problems when the inter-agent communication is restricted. We refer to such estimators as *single time-scale*, shown in Fig. 1-right. Next, we seek a characterization of the single time-scale estimator when the dynamics are arbitrary, i.e., potentially unstable (Tyukin et al., 2007; Urry, 2002); linearized models of fluid dynamics, global waves, or global fluids can also be cast as relevant unstable systems (Urry, 2002). In particular, this paper provides a simple *scalar-gain estimator* where the neighboring observations and the prior estimates are weighted by a

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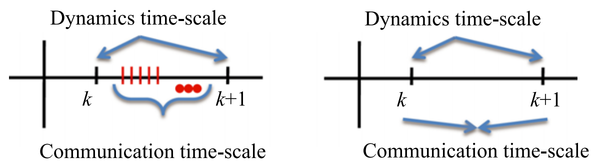

Fig. 1. (Left) Average-consensus; (Right) Single time-scale.

scalar parameter, $\alpha \in \mathbb{R}$. This scalar approach is non-trivial as it: (i) provides a benchmark for the arbitrary (matrix) gain parameters; (ii) is practically simpler to implement; and, (iii) results in closed-form expressions—e.g., consider the scalar-gain average-consensus in Xiao and Boyd (2004).

In the context of scalar-gain estimators, we introduce the notion of *Scalar Tracking Capacity* (STC). The STC is a positive real-number such that every dynamical system whose 2-norm is strictly less than the STC can be estimated with bounded error with the proposed estimator. On the other hand, for every $\|A\|_2 > STC$, we show that there exists a system matrix, A, that cannot be estimated. We provide the closed-form capacity expression along with the optimal scalar-gain and study the estimator performance as a function of the STC. We also formulate some relevant properties of the scalar-gain.

We now describe the rest of the paper. Preliminaries, notation, and the single time-scale estimation are presented in Section 2. Section 3 defines the Scalar Tracking Capacity, and further analyzes the properties and performance of the scalar-gain estimator. Finally, Section 4 provides simulations, and Section 5 concludes the paper.

2. Problem formulation

In this section, we provide notation and the single time-scale, scalar-gain estimator.

2.1. System dynamics and communication graph

The social phenomenon is modeled as a discrete-time, linear dynamical system, perhaps after linearization and discretization, with observations distributed over N agents in the social network[2]:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \mathbf{v}_k, \quad k \geq 0, \tag{1}$$

$$\mathbf{y}_k^i = H_i\mathbf{x}_k + \mathbf{r}_k^i, \quad i = 1, \dots, N, \tag{2}$$

where $\mathbf{x}_k \in \mathbb{R}^n$, $(n > 1)$, is the state-vector, A is a *potentially unstable* system matrix, $\mathbf{v}_k$ is the system noise, $\mathbf{y}_k^i \in \mathbb{R}^{p_i}$ is observation at the ith agent, $H_i \in \mathbb{R}^{p_i \times n}$ is the local observation matrix, and $\mathbf{r}_k^i$ is the local observation noise. We assume the standard assumptions of Gaussianity and independence on the noise variables. The agent observations can be collected to form a global observation, i.e., $\mathbf{y}_k = H\mathbf{x}_k + \mathbf{r}_k$, where $\mathbf{y}_k$, H, and $\mathbf{r}_k$ are collections of $\mathbf{y}_k^i$'s, H_i's, and $\mathbf{r}_k^i$'s, respectively. We assume that the dynamical system, Eqs. (1)–(2), is globally observable in one time-step, i.e., the matrix $\sum_{i=1}^N H_i^\top H_i$ is invertible; any strict subset of agents is not necessarily observable. The stability of the state dynamics is characterized in terms of the induced 2-norm of the system matrix, A, i.e., $a \triangleq \|A\|_2 = \sqrt{\gamma_n}$, where $0 \leq \gamma_1 \leq \cdots \leq \gamma_n$ are the eigenvalues of the symmetric positive (semi) definite (PSD) matrix $A^\top A$.

The interactions among the agents in the social network are modeled as an undirected and connected graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of interconnections among the agents. The neighborhood at the ith agent is defined as $\mathcal{N}_i \triangleq \{j \mid (i, j) \in \mathcal{E}\}$ with $(i, i) \notin \mathcal{E}$. Letting $\text{Adj}(\mathcal{G})$ to denote the corresponding adjacency matrix and $\text{Deg}(\mathcal{G})$ to denote the degree matrix, the graph Laplacian L is defined as $L = \text{Deg}(\mathcal{G}) - \text{Adj}(\mathcal{G})$. We also assume $\mathcal{A} = \text{Adj}(\mathcal{G}) + I_N$.

2.2. Scalar-gain estimator

Given the system dynamics in Eq. (1) with observations in Eq. (2) and the agent communication over $\mathcal{G}$; the goal of this paper is to design a single time-scale estimator of $\mathbf{x}_k$ as motivated in Fig. 1-right. Extension to unstable vector dynamics has the following challenges:

(i) since $\mathbf{x}$ is *not static* and evolves over time, the estimator gain cannot be chosen to go to zero as considered in Kar et al. (2012) and related work;
(ii) since $\mathbf{x}$ is *not scalar*, any agent i is not necessarily observable since $\sum_{j \in \{i\} \cup \mathcal{N}_i} H_j^\top H_j$ may not be invertible; and,
(iii) since the system is *not neutrally-stable*, i.e., $\|A\|_2 > 1$, the collaboration over $\mathcal{G}$ may not be fast enough to track the system evolution.

To overcome these, we consider the following estimator:

$$\widehat{\mathbf{x}}_{k+1}^i = A\widehat{\mathbf{x}}_k^i - \alpha A \sum_{j \in \mathcal{N}_i} \left(\widehat{\mathbf{x}}_k^i - \widehat{\mathbf{x}}_k^j - H_j^\top \left(\mathbf{y}_k^j - H_j \widehat{\mathbf{x}}_k^i \right) \right), \tag{3}$$

where $\widehat{\mathbf{x}}_{k+1}^i \in \mathbb{R}^n$ is the estimate of $\mathbf{x}_{k+1}$ at agent i and time $k+1$ and $\alpha \in \mathbb{R}$ is a scalar-gain.

Motivation: It can be shown that when $\mathbf{x}_k \in \mathbb{R}$ and $A = 1$, Eq. (3) reduces to the scalar estimator in Acemoglu et al. (2008). Additionally, when $A = I_n$ ($n \times n$ identity, $n > 1$), and $\mathbf{v}_k = \mathbf{0}$, Eq. (3) reduces to the static scenario in Kar et al. (2012), while the static parameter estimation, *structurally* similar to Eq. (3), has been considered in DeMarzo, Vayanos, and Zwiebel (2003), Golub and Jackson (2010), and Lopes and Sayed (2008). In particular, without $\sum_{j \in \mathcal{N}_i} \widehat{\mathbf{x}}_k^j$ in Eq. (3), agent i's estimate does not include non-neighboring observations. Since the estimator is single time-scale, non-neighboring observations *cannot* be obtained at agent i by implementing observation fusion, as in Olfati-Saber (2005); $\sum_{j \in \mathcal{N}_i} \widehat{\mathbf{x}}_k^j$ ensures that non-neighboring observations travel within the network. Finally, note that the multiplier $H_j^\top$ appears with the innovation term, $(\mathbf{y}_k^j - H_j\widehat{\mathbf{x}}_k^i)$, because each neighbor's observation, $\mathbf{y}_k^j$, is of different length, p_j; multiplying a $p_j \times 1$ vector, $\mathbf{y}_k^j$, with $n \times p_j$ matrix, $H_j^\top$, ensures that the innovations, of length $n \times 1$, can be added.

The goal of this paper is to characterize the interplay between the *system instability*, $\|A\|_2$, and *information mixing* supported by the graph, $\mathcal{G}$. The formulation here is restricted to a scalar-gain, $\alpha \in \mathbb{R}$, as a function of graph and observation parameters such that the estimation error is bounded. Studying the scalar case provides an intuition towards designing arbitrary (matrix) gains and further results in (benchmark) closed-form expressions.

Estimation error: Let $\mathbf{e}_k^i \triangleq \widehat{\mathbf{x}}_k^i - \mathbf{x}_k$ be the local error at agent i concatenated in a vector, $\mathbf{e}_k$, i.e., the network error process. Let

$$D_H \triangleq \text{blockdiag}\left[\sum_{j \in \mathcal{N}_1} H_j^\top H_j, \ddots, \sum_{j \in \mathcal{N}_N} H_j^\top H_j \right], \tag{4}$$

then the network error process can be written as

$$\mathbf{e}_{k+1} = P\mathbf{e}_k + \mathbf{u}_k, \tag{5}$$

where $P \triangleq (I_N \otimes A)(I_{nN} - \alpha Q)$, the matrix, $Q \triangleq (L \otimes I_n + D_H)$, can be verified to be symmetric, Positive Semi-Definite (PSD), and $\mathbf{u}_k \triangleq \boldsymbol{\phi}_k - \mathbf{1}_N \otimes \mathbf{v}_k$, and $\boldsymbol{\phi}_k \triangleq \alpha(I_N \otimes A) \left[\sum_{j \in \mathcal{N}_1} \mathbf{r}_k^{jT} H_j \quad \dots \quad \sum_{j \in \mathcal{N}_N} \mathbf{r}_k^{jT} H_j \right]^\top$.

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[2] See Friedkin (1998), Newman et al. (2006), Urry (2002) and references within for details on social network modeling and corresponding phenomena of interest.