Automatica 48 (2012) 1489–1501

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Numerical solutions of optimal risk control and dividend optimization policies under a generalized singular control formulation[☆]

Zhuo Jin[a,1], G. Yin[b], Chao Zhu[c]

[a] Centre for Actuarial Studies, Department of Economics, The University of Melbourne, VIC 3010, Australia

[b] Department of Mathematics, Wayne State University, Detroit, MI 48202, United States

[c] Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, United States

ARTICLE INFO

Article history:
Received 1 September 2011
Received in revised form
19 December 2011
Accepted 9 January 2012
Available online 22 June 2012

Keywords:
Singular control
Dividend policy
Markov chain approximation
Numerical method
Reinsurance
Regime switching

ABSTRACT

This paper develops numerical methods for finding optimal dividend pay-out and reinsurance policies. A generalized singular control formulation of surplus and discounted payoff function is introduced, where the surplus is modeled by a regime-switching process subject to both regular and singular controls. To approximate the value function and optimal controls, Markov chain approximation techniques are used to construct a discrete-time controlled Markov chain. The proofs of the convergence of the approximation sequence to the surplus process and the value function are given. Examples of proportional and excess-of-loss reinsurance are presented to illustrate the applicability of numerical methods.

1. Introduction

To design optimal risk controls for a financial corporation has drawn increasing attention since the introduction of the classical collective risk model in Lundberg (1903), where the probability of ruin was considered as a measure of risk. Realizing that the surplus reaching arbitrarily high and exceeding any finite level are not realistic in practice, Bruno de Finetti proposed a dividend optimization problem in De Finetti (1957). Instead of considering the safety aspect (ruin probability), aiming at maximizing the expected discounted total dividends until lifetime ruin, he showed that the optimal dividend strategy is a barrier strategy under the assumption that the surplus process follows a simple random walk. Since then, many researchers have analyzed this problem under more realistic assumptions and extended its range of applications. Some recent work can be found in Asmussen and Taksar (1997), Choulli, Taksar, and Zhou (2001) and Gerber and Shiu (2004) and references therein. To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risk. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Proportional reinsurance is one type of reinsurance policy. Within this scheme, the reinsurance company covers a fixed percentage of losses. The other type of reinsurance policy is nonproportional reinsurance. The most common nonproportional reinsurance policy is the so-called excess-of-loss reinsurance, within which the cedent (primary insurance carrier) will pay all of the claims up to a pre-given level of amount (termed retention level). The comparison of these two types of reinsurance can be found in Asmussen, Høgaard, and Taksar (2000). In this paper, we consider both of these reinsurance policies and provide numerical solutions of the corresponding Markovian regime-switching models.

Let $u(t)$ be an exogenous retention level, which is a control chosen by the insurance company representing the reinsurance policy. In a Cremér–Lundberg model, claims arrive according to a Poisson process with rate β. Let Y_i be the size of the ith claim. The Y_i's are independent and identically distributed (i.i.d.) random variables. Let Y_i^u be the fraction of the claims hold by the cedent.

☆ The research of G. Yin was supported in part by the National Science Foundation under DMS-0907753, and in part by the Air Force Office of Scientific Research under FA9550-10-1-0210. The research of Chao Zhu was supported in part by the National Science Foundation under DMS-1108782, and in part by a grant from the UWM Research Growth Initiative, and City University of Hong Kong (SRG) 7002677. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Qing Zhang under the direction of Editor Berç Rüstem.

E-mail addresses: zjin@unimelb.edu.au (Z. Jin), gyin@math.wayne.edu (G. Yin), zhu@uwm.edu (C. Zhu).

1 Tel.: +61 3 8344 4655; fax: +61 3 8344 6899.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.05.039

The insurer selects the time and the amount of dividends to be paid out to the policyholders. Let $X(t)$ denote the controlled surplus of an insurance company at time $t \geq 0$. Throughout this paper, we only consider cheap reinsurance, where the safety loading for the reinsurer is the same as that for the cedent. The numerical scheme and the convergence proofs are also applicable to more general reinsurance problems. By using the techniques of diffusion approximation applied to the Cremér–Lundberg model, the surplus process satisfies

$$\begin{cases} dX(t) = \beta E[Y_i^u]dt + \sqrt{\beta E[(Y_i^u)^2]}dw(t), \\ X(0^-) = x, \end{cases} \tag{1.1}$$

where $w(t)$ is a standard Brownian motion. In the case of proportional reinsurance, $Y_i^u = uY_i$. Thus, following (1.1), the surplus is given by

$$\begin{cases} dX(t) = \beta u(t)E[Y_i]dt + u(t)\sqrt{\beta E[Y_i^2]}dw(t), \\ X(0^-) = x. \end{cases} \tag{1.2}$$

In the case of excess-of-loss reinsurance, $Y_i^u = Y_i \wedge u$ with the retention level u. We have

$$E[Y^u] = \int_0^u \bar{F}(x)dx, \qquad E[(Y^u)^2] = \int_0^u 2x\bar{F}(x)dx \tag{1.3}$$

where $\bar{F}(x) = P(Y_i > x)$. The stochastic differential equation of the surplus process follows

$$\begin{cases} dX(t) = \beta \displaystyle\int_0^{u(t)} \bar{F}(x)dxdt + \left[\beta \int_0^{u(t)} 2x\bar{F}(x)dx\right]^{\frac{1}{2}} dw(t), \\ X(0^-) = x. \end{cases} \tag{1.4}$$

A common choice of the formulation of the problem is to maximize the total expected discounted value of all dividends until lifetime ruin; see Gerber and Shiu (2006) and Jin, Yin, and Yang (2011). Let

$$\tau := \inf\{t > 0 : X(t) \notin G\} \tag{1.5}$$

be the ruin time, where $G = (0, \infty)$ is the domain of the surplus. Denote by $r > 0$ the discounting factor, and by $Z(t)$ the total dividends paid out up to time t. Our goal is to maximize

$$E \int_0^{\tau} e^{-rt} dZ(t). \tag{1.6}$$

Some "bequest" functions and more complicated utility functions are added to the payoff functions in Browne (1995, 1997). In this paper, we treat payoff functions that are more general and complex than those given in (1.6) or Browne (1995, 1997); our proposed numerical methods are easily implementable.

A dividend strategy $Z(\cdot)$ is an $\mathcal{F}_t$-adapted process $\{Z(t) : t \geq 0\}$ corresponding to the accumulated amount of dividends paid up to time t such that $Z(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous with left limits. In general, a dividend process is not necessarily absolutely continuous. In fact, dividends are not usually paid out continuously in practice. For instance, insurance companies may distribute dividends on discrete time intervals resulting in unbounded payment rate. In such a scenario, the surplus level changes drastically on a dividend payday. Thus abrupt or discontinuous changes occur due to "singular" dividend distribution policy. Together with proportional or excess-of-loss reinsurance policy, this gives rise to a mixed regular–singular stochastic control problem.

Empirical studies indicate in particular that traditional surplus models fail to capture more extreme price movements. To better reflect reality, much effort has been devoted to producing better models. One of the recent trends is to use regime-switching models. Hamilton (1989) introduced a regime-switching time series model, whereas recent work on risk models and related issues can be found in Asmussen (1989) and Yang and Yin (2004). In Wei, Yang, and Wang (2010), the optimal dividend and proportional reinsurance strategies under utility criteria were studied for the regime-switching compound Poisson model by using the methods of the classical and impulse control theory. Optimal dividend strategies under a regime-switching diffusion model was studied in Sotomayor and Cadenillas (2011). A comprehensive study of switching diffusions with "state-dependent" switching is in Yin and Zhu (2010).

In this work, we model the surplus process by a regime-switching diffusion; we also consider reinsurance and dividend payment policies as regular and singular stochastic controls. Our goal is to maximize the expected total discounted payoff until ruin; see (2.2) for details. The model we consider appears to be more versatile and realistic than the classical compound Poisson or diffusion models. To find the optimal reinsurance and dividend pay-out strategies, one usually solves a so-called Hamilton–Jacobi–Bellman (HJB) equation. However, in our work, due to the regime switching and the mixed regular and singular control formulation, the HJB equation is in fact a coupled system of nonlinear quasi-variational inequalities (QVIs). A closed-form solution is virtually impossible to obtain. A viable alternative is to employ numerical approximations. In this work, we adapt the Markov chain approximation methodology developed by Kushner and Dupuis (2001). To the best of our knowledge, numerical methods for singular controls of regime-switching diffusions have not been studied in the literature to date. Even for singular controlled diffusions without regime switching, the related results are relatively scarce; Budhiraja and Ross (2007) and Kushner and Martins (1991) are the only papers that carry out a convergence analysis using weak convergence and relaxed control formulation of numerical schemes for singular control problems in the setting of Itô diffusions. We focus on developing numerical methods that are applicable to mixed regular and singular controls for regime-switching models. Although the primary motivation stems from insurance risk controls, the techniques and the algorithms suggested are applicable to other singular control problems. It is also worth mentioning that the Markov chain approximation method requires little regularity of the value function and/or analytic properties of the associated systems of HJB equations and/or QVIs. The numerical implementation can be done using either value iterations or policy iterations.

The rest of the paper is organized as follows. A generalized formulation of optimal risk control and dividend policies and assumptions are presented in Section 2. The two most common reinsurance strategies (proportional reinsurance and excess-of-loss reinsurance) are covered in our study. Section 3 deals with the numerical algorithm of Markov chain approximation method. The regular control and the singular control are well approximated by the approximating Markov chain and the dynamic programming equation are presented. Section 4 studies the convergence of the approximation scheme. The technique of "rescaling time" is introduced and the convergence theorems are proved. Two classes of numerical examples are provided in Section 5 to illustrate the performance of the approximation method. Finally, some additional remarks are provided in Section 6.

2. Formulation

In this section, we introduce a dynamic system to describe the surplus processes with reinsurance and dividend payout strategies with Markov regime switching. Let $X(t)$ denote the controlled surplus of an insurance company at time $t \geq 0$. Denote by $u(t)$ and $Z(t)$ the dynamic reinsurance policy at time t and the total