Automatica 48 (2012) 1707–1714

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Two families of semiglobal state observers for analytic discrete-time systems[☆]

Alfredo Germani [a,b], Costanzo Manes [a,c,1]

[a] Department of Electrical and Information Engineering, University of L'Aquila, Via G. Gronchi, 18, 67100 LAquila, Italy
[b] Università Campus Bio-Medico of Rome, Via Álvaro del Portillo, 21, 00128 Roma, Italy
[c] Istituto di Analisi dei Sistemi ed Informatica del CNR "A. Ruberti", Viale Manzoni 30, 00185 Roma, Italy

ARTICLE INFO

Article history:
Received 17 February 2011
Received in revised form
7 October 2011
Accepted 7 January 2012
Available online 26 June 2012

Keywords:
State observers
Nonlinear systems
Discrete-time systems
Analytic approximations

ABSTRACT

Two families of observers for discrete-time nonlinear systems are presented in this paper, whose design is based on the Taylor approximation of the inverse of the observation map. Semiglobal convergence results are provided under the assumption that the system observation map is a globally analytic diffeomorphism. The performances of the observers in the two families are compared both from theoretical and practical points of view.

1. Introduction

The problem of state reconstruction for nonlinear systems from input and output measurements has been widely investigated in the literature, and many techniques exist for the design of asymptotic state observers. One method consists in finding a nonlinear change of coordinates and an output injection that recast the system into some canonical form, suitable for a linear observer design. In the discrete-time framework, first papers dealing with this approach are Lee and Nam (1991) and Lin and Byrnes (1995), where autonomous systems are only considered. More recent papers are Xiao, Kazantzis, Kravaris, and Krener (2003) and Xiao (2006). The case of systems with input is considered by Besançon and Bornard (1995), Besançon, Hammouri, and Benamor (1998), Califano, Monaco, and Normand-Cyrot (2003, 2009). In general, the appropriate coordinate transformation exists under quite restrictive conditions and its computation is a very difficult task. An interesting technique for the construction of observers with linear error dynamics for systems admitting a differential/difference representation is in Monaco, Normand-Cyrot, and Barbot (2007). Another approach exploits dynamic inversion of suitably defined *observation maps* to achieve asymptotic state reconstruction without the need of any coordinate transformation (Ciccarella, Dalla Mora, & Germani, 1993, 1995). Local convergence of these observers is proved under standard Lipschitz assumptions. The use of the Extended Kalman Filter as a local observer has been investigated in Boutayeb and Aubry (1999), Boutayeb, Rafaralahy, and Darouach (1997) and Reif and Unbehauen (1999), while in Germani and Manes (2008) the convergence of the Polynomial Extended Kalman Filter (Germani, Manes, & Palumbo, 2005), when used as an observer, is studied. Observers for the case of nonlinear systems with linear measurements are considered in Abbaszadeh and Marquez (2008), Boutayeb and Darouach (2000) and Ibrir (2007). An H_∞ observer design approach is followed by Zemouche, Boutayeb, and Bara (2008), Zemouche and Boutayeb (2009a,b). Another approach is the Moving Horizon Estimation technique, as in Kang (2006), which allows to consider also uncertainties and disturbances, as in Alessandri, Baglietto, and Battistelli (2008).

This paper presents two families of *semiglobal* observers, based on high order Taylor approximations of the inverse of the observation map, that improve the *local* observers in Ciccarella et al. (1993, 1995), based on the first order Taylor approximation. The degree ν of the approximating polynomial defines the order of the observer in the families. Our approach takes inspiration from Germani, Manes, Palumbo, and Sciandrone (2006), where a root-finding method has been developed by suitably exploiting Taylor polynomials of degree $\nu > 1$ to get higher convergence rates than the Newton–Raphson method. The main feature of the presented observer families is that, *for any* given bound on

☆ This work is supported by *MIUR* (Italian Ministry of Education, University, and Research) grant PRIN2009 N. 2009J7FWLX_002, and by *CNR* (National Research Council of Italy). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Henri Huijberts under the direction of Editor Andrew R. Teel.

E-mail addresses: alfredo.germani@univaq.it (A. Germani), costanzo.manes@univaq.it (C. Manes).

1 Tel.: +39 0862 434404; fax: +39 0862 434403.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.05.030

the initial observation error, the degree ν can be chosen large enough to guarantee the convergence of the observation error to zero at *any desired* exponential rate (semiglobal exponential convergence).

The paper is organized as follows. Preliminary definitions and notations are provided in Section 2. The formulas of the Taylor polynomial expansion of the inverse of the observation map are presented in Section 3. Section 4 presents two families of state observers and the convergence theorems for the case of unforced systems. The case of systems with input is discussed in Section 5. Simulation results and conclusions follow.

2. Preliminaries

This paper deals with the problem of state-observers design for nonlinear discrete-time systems of the type

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t), u(t)), \end{aligned} \qquad t \in \mathbb{Z}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the unknown state, $u(t) \in \bar{U} \subseteq \mathbb{R}$ is a known input, and $y(t) \in \mathbb{R}$ is the measured output. $f : \mathbb{R}^n \times \bar{U} \mapsto \mathbb{R}^n$ is the one-step state transition function, and $h : \mathbb{R}^n \times \bar{U} \mapsto \mathbb{R}$ is the output function. Both functions f and h are assumed to be analytic.

The observer design methodology presented in this paper relies on the so called *observation map*, that is the square function that transforms the system state at a given time t into the output sequence in the interval $[t, t+n) \subset \mathbb{Z}$. The formal definition of the observation map requires the introduction of some notations. Throughout the paper, for a given vector $V \in \bar{U}^r \subseteq \mathbb{R}^r$, the symbols $V_{[1:k]}$, $V_{[k]}$ and $V_{[r-k+1:r]}$, with $k < r$, will denote the first k components, the k-th component and last k components of V, respectively. This allows to define the r-steps state transition functions $f^r(x, V)$, with $r \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $V \in \bar{U}^r$, as

$$\begin{aligned} &f^1(x, V) = f(x, V), \quad [\text{and } f^0(x) = x,] \\ &f^r(x, V) = f(f^{r-1}(x, V_{[2:r]}), V_{[1]}), \quad r > 1. \end{aligned} \tag{2}$$

Alternatively, $f^r(x, V) = f^{r-1}(f(x, V_{[r]}), V_{[1:r-1]})$.

The symbol $h \circ f^{r-1}$ will denote the function defined as

$$\begin{aligned} &h \circ f^0(x, V) = h(x, V), \quad V \in \bar{U}, \\ &h \circ f^{r-1}(x, V) = h(f^{r-1}(x, V_{[2:r]}), V_{[1]}), \quad V \in \bar{U}^r. \end{aligned} \tag{3}$$

The n functions $h \circ f^{r-1}(x)$, $r = 1, \ldots, n$, can be stacked into a square map $z = \Phi(x; V)$, with $V \in \bar{U}^n$, as follows:

$$\Phi(x; V) = \begin{bmatrix} h \circ f^{n-1}(x, V_{[1:n]}) \\ \vdots \\ h \circ f^1(x, V_{[n-1:n]}) \\ h(x, V_{[n]}) \end{bmatrix}. \tag{4}$$

Given the input and output sequences $u(t)$ and $y(t)$, let us define the vectors $U_t \in \bar{U}^n$ and $Y_t \in \mathbb{R}^n$ as

$$Y_t = \begin{bmatrix} y(t+n-1) \\ \vdots \\ y(t+1) \\ y(t) \end{bmatrix}, \qquad U_t = \begin{bmatrix} u(t+n-1) \\ \vdots \\ u(t+1) \\ u(t) \end{bmatrix}, \tag{5}$$

so that the following relation holds for any $t \in \mathbb{Z}$:

$$Y_t = \Phi(x(t); U_t). \tag{6}$$

The function $z = \Phi(x; V)$ defined in (4) is a square map from $x \in \mathbb{R}^n$ to $z \in \mathbb{R}^n$, where $V \in \bar{U}^n$ is a vector of known parameters. If such a map is invertible, then the state reconstruction from the knowledge of the input and output sequences (Y_t and U_t) is theoretically possible. For this reason, the following definitions are given.

Definition 1. The map $\Phi : \mathbb{R}^n \times \bar{U}^n \mapsto \mathbb{R}^n$ defined in (4) is called the *observation map* of the system (1), and its Jacobian $\nabla_x \Phi(x, V)$ is called the *observability matrix*.

Definition 2. The nonlinear system (1) with $u(t) \in \bar{U}$, is said to be uniformly *observable* in a subset $\Omega \subseteq \mathbb{R}^n$ if its observation map (4) is invertible in Ω for any $V \in \bar{U}^n$. If $\Omega = \mathbb{R}^n$, then the system (1) is said to be *globally observable*. If $\bar{U} = 0$, the system is said to be drift-observable.

The inverse of the observation map is symbolically written as $x = \Phi^{-1}(z, V)$.

Remark 1. The invertibility of $\Phi(x; V)$ may depend on the set $\bar{U}$ of admissible inputs. When $\bar{U} = \mathbb{R}$, uniform observability in $\Omega \subset \mathbb{R}^n$ is equivalent to *observability for any input* (see Gauthier, Hammouri, & Othman, 1992, for continuous-time systems). This is a rather strong property, even stronger when $\Omega = \mathbb{R}^n$ (global uniform observability), because in general the inverse of a nonlinear map is only locally well-defined, and often admits bifurcation points (see e.g. Barbot, Belmouhoub, & Boutat-Baddas, 2006). However, *uniform observability for any input in a subset $\bar{U} \subset \mathbb{R}$* can be a much weaker property, because $\bar{U}$ can be small enough to keep out *bad inputs*. In Dalla Mora, Germani and Manes (2000), it is shown that any drift-observable system admits a bounded set $\bar{U}$ such that the system is uniformly observable for any $u(t) \in \bar{U}$.

When the uniform observability assumption for any input in $\bar{U}$ is satisfied, the presence of the parameter V in the observation map (4) and in the r-steps transition functions (2) and output functions (3) does not add any theoretical complication to the state reconstruction schemes here presented. Thus, in order to have simpler notations, the case of unforced discrete-time systems is considered at first:

$$\begin{aligned} x(t+1) &= f(x(t)), \\ y(t) &= h(x(t)), \end{aligned} \qquad t \in \mathbb{Z}, \tag{7}$$

so that $f^0(x) = x$ and

$$\begin{aligned} &f^{r+1}(x) = (f \circ f^r)(x) = f(f^r(x)), \quad r \geq 0, \\ &h \circ f^r(x) = h(f^r(x)). \end{aligned} \tag{8}$$

The observation map takes the simpler form

$$\Phi(x) = [h \circ f^{n-1}(x) \ \cdots \ h \circ f^1(x) \ h(x)]^T, \tag{9}$$

and the output sequence is a function of the state only

$$Y_t = \Phi(x(t)). \tag{10}$$

Definition 3. The nonlinear system (7) is said to be *observable* in a subset $\Omega \subseteq \mathbb{R}^n$ if its observation map (9) is invertible in Ω. If $\Omega = \mathbb{R}^n$, then the system (7) is said to be *globally observable*.

By Definition 3, a system is observable if for any $t \in \mathbb{Z}$ the output sequence in the interval $[t, t+n)$ univocally determines the state x at time t, *formally* expressed as a function of the inverse map

$$x(t) = \Phi^{-1}(Y_t). \tag{11}$$

In (11), the *current* state $x(t)$ is written as a function of *future* observations. The causal computation of $x(t)$ as a function of current and past observations Y_{t-n+1} can be made in two steps (*ideal exact state reconstruction*):

1a. compute the state at time $t-n+1$ as

$$x(t-n+1) = \Phi^{-1}(Y_{t-n+1}), \tag{12}$$

2a. compute the current state $x(t)$ as

$$x(t) = f^{n-1}(x(t-n+1)). \tag{13}$$