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Brief paper

Switching control of closed quantum systems via the Lyapunov method[☆]

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ABSTRACT

In this paper, we consider the state transfer problem for closed quantum systems under a degenerate case, where the linearized system around the target state is not controllable. It is known that the traditional Lyapunov control methods may fail to guarantee the convergence to the target state under the degenerate case. Hence, we propose to use multiple Lyapunov functions and design a switching control strategy to achieve more accurate state transfer. It is shown that the system can converge to the intersection of invariant sets including the target state. The explicit analysis of the convergence is provided to design the switching law. Moreover, the effectiveness of open-loop Lyapunov control is discussed. Simulation studies are presented to show the improved control performance.

1. Introduction

Driven by recent technological developments and its promise in a wide variety of applications, such as quantum computation (Mitra & Rabitz, 2003; Nielsen & Chuang, 2000), NMR (D'Alessandro, 2007), quantum chemistry (Tersigni, Gaspard, & Rice, 1990), and quantum optics (Nurdin, James, & Petersen, 2009), quantum control has been attracting more and more research attention these last few years, e.g., controllability of quantum systems using Lie algebra (D'Alessandro, 2010), optimal control (Ho & Rabitz, 2010), Lyapunov control (Altafini, 2007; Beauchard, Coron, Mirrahimi, & Rouchon, 2007; Grivopoulos & Bamieh, 2003; Kuang & Cong, 2008; Mirrahimi, Rouchon, & Turinici, 2005; Wang & Schirmer, 2010a,b), measurement assistant control (Pechen, Il'in, Shuang, & Rabitz, 2006), closed-loop learning control (Dong, Chen, Tarn, Pechen, & Rabitz, 2008), and closed-loop feedback control (Mirrahimi & Handel, 2007; Van Handel, Stockton, & Mabuchi, 2005; Wang & Wiseman, 2001; Yamamoto, Tsumura, & Hara, 2007).

One central problem is how to design the control to steer the quantum state to a fixed target state, or track an adiabatic trajectory. Much attention has been focused on the Lyapunov control of closed quantum systems by assuming that the interactions with the environment can be neglected, see Altafini (2007), Beauchard et al. (2007), Grivopoulos and Bamieh (2003), Kuang and Cong (2008), Wang and Schirmer (2010a,b) and Mirrahimi et al. (2005). Since measurements and feedback would lead to more complicated models than Schrödinger equations, a usual practice in Lyapunov-based quantum control is to first obtain a control signal from a simulation study and then apply it to real systems, i.e., open-loop control with precalculated control signals.

A great deal of progress has been made in the conventional Lyapunov method. However, it may fail to achieve the control objective when the linearized system around the target state is not controllable. This case is called a *degenerate case* in the literature, which could happen in practical quantum systems such as the five-level system shown in Ramakrishna, Salapaka, Dahleh, Rabitz, and Peirce (1995) and Tersigni et al. (1990) and the four-level molecular systems in Gross, Neuhauser, and Rabitz (1991) and Phan and Rabitz (1999). To deal with this degenerate case, alternative methods based on an implicit Lyapunov function and trajectory tracking were proposed in Beauchard et al. (2007) and Mirrahimi et al. (2005), respectively. However, the difficulties of these methods lie in the characterization of LaSalle invariant sets and the convergence analysis. To overcome these difficulties, we propose a switching control.

In the past decade, switching control of classical systems has attracted much attention. It can stabilize unstable systems and

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greatly improve the control performance (Baldi, Battistelli, Mosca, & Tesi, 2010; Dong & Sun, 2008; Lin & Antsaklis, 2007; Santarelli, 2011; Ye, 2005). Recently, the switching control has been extended to quantum systems (Coron, Grigoriu, Lefter, & Turinici, 2009; Dong & Petersen, 2010; Khaneja, 2007; Lou, Cong, Yang, & Kuang, 2011; Mirrahimi & Handel, 2007). This motivates us to investigate a more intuitive switching control of closed quantum systems for the degenerate case. The basic idea of the proposed method is based on the combination of the multiple Lyapunov functions and switching control. It is interesting to find that the switching control can achieve the convergence to the intersection of invariant sets, strictly smaller than any of the invariant sets. For the switching control, one main difficulty lies in how to design the switching law. By the explicit convergence analysis, a state-based switching law is designed. Another difficulty is how to guarantee the effectiveness of the switching control in the presence of disturbances induced by the control fields or initial states. By estimating the distance between a nominal system and its perturbed system, we find that the switching control is effective in spite of small disturbances.

The rest of this paper is organized as follows: Section 2 gives the preliminaries on the conventional Lyapunov control of closed quantum systems. In Section 3, the switching control strategy and the implementation procedure are proposed, respectively. Moreover, the performance of open-loop Lyapunov control under disturbances is discussed. Section 4 includes numerical simulations to show the effectiveness and advantage of the proposed method. Finally, concluding remarks are drawn in Section 5.

2. Preliminaries

Consider a finite dimensional closed quantum system, modeled as the following Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi\rangle = \left[H_0 + \sum_{k=1}^{r} H_k u_k(t)\right]|\Psi\rangle, \qquad |\Psi\rangle|_{t=0} = |\Psi_0\rangle, \qquad \| \,|\Psi_0\rangle\, \| = 1, \tag{1}$$

where $\hbar$ is the reduced Planck constant and set to be 1 in this paper. The internal Hamiltonian H_0 and the control Hamiltonian H_k are Hermitian matrices. $u_k \in \mathbb{R}$ is the control, $k = 1, 2, \ldots, r$. We denote $|\phi_i\rangle$ to be the eigenstate of H_0 associated with the eigenvalue λ_i, $i = 1, 2, \ldots, n$. In this eigenbasis, a superposition state can be described as $|\Psi(t)\rangle = \sum_{i=1}^{n} c_i(t)|\phi_i\rangle$, satisfying $\sum_{i=1}^{n} |c_i|^2 = 1$. It evolves on the $(n-1)$-dimensional complex unit sphere $\mathbb{S}^{n-1} := \{|\Psi(t)\rangle \in \mathbb{C}^n : \| \,|\Psi(t)\rangle\, \| = 1\}$. The objective of this paper is to design the control u_k to drive the quantum system from an initial state $|\Psi_0\rangle$ to a prescribed target state $|\phi_f\rangle$, which is usually chosen to be an eigenstate of H_0.

In the classical Lyapunov control of quantum systems, the following two typical Lyapunov functions are defined,

$$V_1 = \langle\Psi|P|\Psi\rangle, \qquad V_2 = 1 - |\langle\Psi|\phi_f\rangle|^2. \tag{2}$$

V_1 is based on the expectation of a Hermitian operator P, and the construction method of V_1 can be found in Kuang and Cong (2008). Here, we construct $P = \sum_{i=1}^{n} p_i|\phi_i\rangle\langle\phi_i|$, where p_i is the eigenvalue of P corresponding to the eigenstate $|\phi_i\rangle$, $i = 1, 2, \ldots, n$. From $|\Psi_0\rangle$, if the eigenvalue of P corresponding to $|\phi_f\rangle$ is minimal and V_1 decreases continuously under the control u_k, then it will be possible to drive the system to $|\phi_f\rangle$. By constructing a different operator P, where the minimal eigenvalue is associated with a different eigenstate of H_0, we can change the control result. This idea will be used to construct the Hermitian operator P based on the magnitude order of its eigenvalues. V_2 is the Lyapunov function based on the Hilbert–Schmidt distance between $|\Psi\rangle$ and $|\phi_f\rangle$, denoted as $\mathrm{dist}(|\Psi\rangle, |\phi_f\rangle)$. The following two kinds of control are designed to ensure V_1 and V_2 to be nonincreasing, respectively,

$$u_{1k} = -K_{1k} f_{1k}(i\langle\Psi|[H_k, P]|\Psi\rangle), \qquad u_{2k} = K_{2k} f_{2k}(\Im(e^{i\angle\langle\Psi|\phi_f\rangle}\langle\phi_f|H_k|\Psi\rangle)), \tag{3}$$

where $k = 1, 2, \ldots, r$, $\Im(\cdot)$ is the imaginary part of a complex number, and $\angle\langle\Psi|\phi_f\rangle$ denotes the phase between $|\Psi\rangle$ and $|\phi_f\rangle$. The control gains K_{1k} and K_{2k} are positive, and f_{1k} and f_{2k} are functions passing through the origin and satisfying $f_{ik}(x)x \geq 0$, $i = 1, 2$. The invariant sets can be characterized by the LaSalle invariance principle (Kuang & Cong, 2008; Mirrahimi et al., 2005). Assuming that

$$\text{(i) } [H_0, P] = 0, \qquad \text{(ii) } \omega_{ij} \neq \omega_{lm}, \quad (i, j) \neq (l, m), \qquad \text{(iii) } p_i \neq p_j, \quad i \neq j, \tag{4}$$

where $\omega_{ij} = \lambda_i - \lambda_j$, $i, j, l, m = 1, 2, \ldots, n$, the control u_{1k} drives the system state to the LaSalle invariant set $S_1 \cap \mathbb{S}^{n-1}$

$$S_1 = \{|\Psi\rangle : \langle\phi_i|H_k|\phi_j\rangle\langle\phi_i|\Psi\rangle\langle\Psi|\phi_j\rangle = 0, \quad i \neq j, \quad k = 1, 2, \ldots, r\}. \tag{5}$$

The control u_{2k} steers the system state to the LaSalle invariant set $S_2 \cap \mathbb{S}^{n-1}$

$$S_2 = \{|\Psi\rangle : |\Psi\rangle = \Sigma_\alpha c_\alpha|\phi_\alpha\rangle,\ \langle\phi_f|H_k|\phi_\alpha\rangle = 0, \quad k = 1, 2, \ldots, r\}. \tag{6}$$

When the linearized system around the target state is not controllable, it is difficult for the classical Lyapunov control to achieve the state transfer, which will be simulated in Section 4. It should be noted that S_1 and S_2 contain different limit points. Moreover, the Hermitian operator P in V_1 can be constructed flexibly according to practical requirements. This motivates us to consider a switching control of quantum systems.

3. Switching control by the Lyapunov method

3.1. Switching control design

For convenience, we rearrange the indices of the eigenstates of H_0. Let $|\phi_n\rangle$ be the target state. The eigenstates $|\phi_i\rangle$ satisfying $\langle\phi_n|H_k|\phi_i\rangle = 0$ are denoted as $|\phi_{l+1}\rangle, \ldots, |\phi_{n-1}\rangle$, $k = 1, 2, \ldots, r$. It is assumed that H_0 is not λ-degenerate, i.e., there does not exist λ such that $|\lambda_i - \lambda| = |\lambda_j - \lambda|$, $\forall i, j \in \{1, 2, \ldots, n\}$. Assume that there exists $k \in \{1, 2, \ldots, r\}$ such that $\langle\phi_i|H_k|\phi_j\rangle \neq 0$, $i \neq j$, except for $i \in \{l+1, \ldots, n-1\}$ and $j = n$. Even with this relatively conservative assumption, the classical Lyapunov control strategies fail to drive the system, as illustrated in Section 4. With control (3), the system can only converge to the invariant set S_1 or S_2. In particular, under the above assumption, S_1 and S_2 can be characterized as

$$S_1 = \{|\Psi\rangle : |\Psi\rangle = c_1|\phi_i\rangle + c_2|\phi_n\rangle, |c_1|^2 + |c_2|^2 = 1, \quad i = l+1, \ldots, n-1\} \cup \{|\phi_1\rangle, \ldots, |\phi_l\rangle\}, \tag{7}$$
$$S_2 = \{|\Psi\rangle : |\Psi\rangle = \Sigma_{i=l+1}^{n} c_i|\phi_i\rangle, \Sigma_{i=l+1}^{n}|c_i|^2 = 1, c_i \in \mathbb{C}\}.$$

Next, we investigate the switching control between u_{1k} and u_{2k} such that the system state is driven to the intersection of S_1 and S_2, $k = 1, 2, \ldots, r$.

Theorem 1. *For system (1), if conditions (4) hold and there exists $k \in \{1, 2, \ldots, r\}$ such that $\langle\phi_i|H_k|\phi_j\rangle \neq 0$ except for $i \in \{l+1, \ldots, n\}$ and $j = n$, the switching control sequence $\{u_{1k}, u_{1i_*k}, u_{2k}, \bar{u}_{1k}\}$ drives the system to the intersection $S_1 \cap S_2$, where u_{1k}, u_{1i_*k} and $\bar{u}_{1k}$ are the control signals based on different Hermitian operators P by assigning different eigenstates associated with the minimal eigenvalue, $k = 1, 2, \ldots, r$.*

Proof. According to the above analysis, we prove the theorem by constructing different Hermitian operators P. When the system