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Brief paper

Leader–follower consensus over numerosity-constrained random networks☆

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ABSTRACT

In this work, we study a discrete-time consensus protocol for a group of agents which communicate over a class of stochastically switching networks inspired by fish schooling. The network model incorporates the phenomenon of numerosity, that plays a prominent role in the collective behavior of animal groups, by defining the individuals' perception of numbers. The agents comprise leaders, which share a common state, and followers, which update their states based on information exchange among neighboring agents. We establish a closed form expression for the asymptotic convergence factor of the protocol, that measures the decay rate of disagreement among the followers' and the leaders' states. Handleable forms of this expression are derived for the physically relevant cases of large networks whose agents are composed of primarily leaders or followers. Numerical simulations are conducted to validate analytical results and illustrate the consensus dynamics as a function of the number of leaders in the group, the agents' persuasibility, and the agents' numerosity. We find that the maximum speed of convergence for a given population can be enhanced by increasing the proportion of leaders in the group or the agents' numerosity. On the other hand, we find that increasing the numerosity has also a negative effect as it reduces the range of agents' persuasibility for which consensus is possible. Finally, we compare the main features of this leader–follower consensus protocol with its leaderless counterpart to elucidate the benefits and drawbacks of leadership in numerosity-constrained random networks.

1. Introduction

Fish schooling is a striking example of collective behavior of social animals, in which complex ordered states emerge from local interactions, see for example (Partridge, 1982). The distinctive characteristics of fish schooling include highly coordinated motion and relatively small inter-fish distance. Such interactions are mediated by the perceptual capabilities of the individual fish, which include vision and sensing of flow, electrical, and chemical signals, as well as psychological factors such as numerosity. Specifically, numerosity limits the perception of exact numbers across species, including fish (Tegeder & Krause, 1995), birds (Ballerini et al., 2008), and humans (Piazza & Izard, 2009). Recent behavioral studies have posited that fish schools are not leaderless societies as previously thought. Leaders are defined to be fish who initiate new directions of locomotion readily taken up by the followers (Krause, Hoare, Krause, Hemelrijk, & Rubenstein, 2000). Leaders are more likely to be larger members in the school, even if boldness and hunger may elicit followership from peers (Krause, Reeves, & Hoare, 1998; Leblond & Reebs, 2006). It is the social feedback between followers and leaders which effectually determines the successful maneuvering of the group as a cohesive unit (Couzin, Krause, Franks, & Levin, 2005; Harcourt, Ang, Sweetman, Johnstone, & Manica, 2009).

Collective behavior has been effectively modeled as a consensus protocol, that is, a distributed algorithm in which individual agents seek agreement on a quantity of interest through iterated negotiations, see for example the reviews (Bertsekas & Tsitsiklis, 1997; Ren & Beard, 2008). Such negotiations are informed by an underlying network structure that determines interactions between individuals and thus controls the information flow. Specifically, the occurrence of an interaction between individuals is modeled as a network edge and individuals are interpreted as the networked nodes. The degree of a node, that is, the number of individuals with whom an agent interacts, describes the number of neighbors that are used for negotiating the quantity of interest. Motivated by our earlier work on fish schooling in annular domains (Abaid & Porfiri, 2010) and breakthroughs in Ballerini et al. (2008), we have introduced a new class of networks, called numerosity-constrained networks, which incorporates the limitation imposed by numerosity in Abaid and Porfiri (2011). Therein, we have explored consensus over stochastically switching numerosity-constrained networks where all agents are identical in that they execute the same updating protocol.

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Consensus over stochastically switching networks is a relatively untapped research topic, despite its potential for application in biological and technological settings, such as animal groups (Ballerini et al., 2008; Couzin et al., 2005; Harcourt et al., 2009; Partridge, 1982) and wireless sensor networks (Erkip, Sendonaris, Stefanov, & Aazhang, 2004). For protocols without leaders, criteria for almost sure convergence are presented in Hatano and Mesbahi (2005) and Huang and Manton (2010), Porfiri and Stilwell (2007), Tahbaz-Salehi and Jadbabaie (2008), Wu (2006) for undirected and directed communication networks, respectively, for mean square convergence in Kar and Moura (2008) only for undirected networks, and for $\mathcal{L}^p$ convergence in Liu, Lu, and Chen (2011). The rate of convergence to consensus for undirected and directed networks is studied in Hatano and Mesbahi (2005), Kar and Moura (2008), Patterson, Bamieh, and Abbadi (2010) and Pereira and Pages-Zamora (2010), Zhou and Wang (2009), respectively.

In this paper, we study a leader–follower consensus protocol over a stochastic numerosity-constrained network in which agents are either informed leaders or naive followers, see for example (Ren & Beard, 2008). Leaders have a common state that is constant over time, while followers update their states over time based on information exchange among both leaders and followers. Consensus problems for groups comprising leaders and followers are studied for example in Hong, Hu, and Gao (2006), Khan, Kar, and Moura (2010), Meng, Ren, Cao, and You (2011), Song, Cao, and Yu (2010), Yu, Chen, and Cao (2010), Zhu and Cheng (2010). These works assume that the underlying network of communication among leaders and followers is either static (Khan et al., 2010; Meng et al., 2011; Song et al., 2010) or varying over time according to a predefined (Hong et al., 2006; Zhu & Cheng, 2010) or state-dependent switching pattern (Yu et al., 2010). Here, we take a different approach as we focus on stochastic leader–follower consensus to establish necessary and sufficient conditions for mean square consentability based on a closed form expression for the rate of convergence to consensus. The derived closed form expression allows us to dissect the effect of the multiple factors determining the feasibility of achieving consensus in leader–follower numerosity-constrained random networks and its performance; these factors include the number of agents, the proportion of leaders/followers, the agents' numerosity, and the agents' persuasibility. The technical approach used in the derivation of such closed form result shares similarities with the methodology presented in Abaid and Porfiri (2011) that is based on the exact computation of the eigenstructure of a higher order state matrix, yet the presence of two types of agents introduces more complex algebraic structures than (Abaid & Porfiri, 2011).

2. Problem statement

We consider a system of N agents, comprising l leaders and f followers, $f, l \in \{1, 2, 3, \ldots\}$, which update their states based on communication over a directed network with a stochastically switching numerosity-constrained topology. Neighbors in the communication network are defined as agents who unidirectionally communicate state information. At time step $k \in \mathbb{Z}^+$, we describe the communication network through the graph Laplacian $L_k \in \mathbb{R}^{N\times N}$, see for example (Bollobas, 1998). According to the numerosity-constrained topology, each agent has $n \in \{1, 2, \ldots, N-1\}$ neighbors chosen with equal probability from all other agents. More specifically, the ith row of L_k has diagonal entry equal to n and offdiagonal entries comprising n "-1's" and $N-n-1$ "0's" and each combination of these $N-1$ entries is equally likely. The stochastic network can be viewed as a sequence of independent identically distributed (IID) matrices $\{L_k\}_{k=0}^{\infty}$ with common random variable L. We notice that L_k has the zero row-sum property $L_k 1_N = 0_N$ and is not necessarily symmetric due to the unidirectional communication among agents.

At time step $k \in \mathbb{Z}^+$, the agents' states are given by the vector $X_k = [x_k^{\mathrm{T}}\ y_k^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^N$, where $x_k \in \mathbb{R}^f$ is the state vector for the followers, $y_k \in \mathbb{R}^l$ is the state vector for the leaders, and superscript T denotes the matrix transpose. The full state vector X_k is updated according to the discrete-time consensus protocol

$$X_{k+1} = (I_N - \mathcal{E} L_k)X_k, \tag{1}$$

with initial condition $X_0 = [x_0^{\mathrm{T}}\ s1_l^{\mathrm{T}}]^{\mathrm{T}}$ for $x_0 \in \mathbb{R}^f$ and $s \in \mathbb{R}$. The matrix $\mathcal{E} \in \mathbb{R}^{N\times N}$ is the diagonal matrix $\mathrm{diag}\left([\varepsilon 1_f^{\mathrm{T}}\ 0_l^{\mathrm{T}}]^{\mathrm{T}}\right)$ with $\varepsilon \in \mathbb{R}^+$, where the operator $\mathrm{diag}(\cdot)$ takes the ith entry of an $N \times 1$ vector to the ith diagonal entry of an $N \times N$ diagonal matrix. In other words, the leaders share common initial states which are not updated in the protocol, while the followers update their states over time according to those of their neighbors, mediated to a weighting parameter ε which describes their persuasibility (Abaid & Porfiri, 2011). We say that agents achieve consensus when the followers take the leaders' state, that is, when $X_k = s1_N$. We note that the state matrix in (1) is not required to be nonnegative, unlike most of state matrices studied in the literature for consensus problems (Hatano & Mesbahi, 2005; Porfiri & Stilwell, 2007; Tahbaz-Salehi & Jadbabaie, 2008, 2010).

In the remainder of the paper, we use the following notation. The vector e_i is the ith column of the identity matrix I_f. The vectorizing function $\mathrm{vec}(\cdot)$ stacks the columns of an $N \times M$ matrix to create an $NM \times 1$ column vector. The Euclidean norm of a vector is denoted by $\|\cdot\|$. The operation $\otimes$ is the Kronecker product. The expected value of a random variable is written $\mathbf{E}[\cdot]$.

3. Analysis

The consensus problem (1) is reduced to the subsystem describing the followers' states x_k since the leaders do not update their state s over time. Specifically, we write the first f rows of L_k as the augmented matrix $[\widehat{L}_k\ \widehat{K}_k]$, where $\widehat{L}_k \in \mathbb{R}^{f\times f}$ and $\widehat{K}_k \in \mathbb{R}^{f\times l}$. From the definition of L_k, $[\widehat{L}_k\ \widehat{K}_k]$ has zero-row sum and $\{\widehat{L}_k\}_{k=0}^{\infty}$ is a sequence of IID random variables, whose common random variable we write as $\widehat{L}$. Using this block structure in (1), we isolate the f-dimensional followers subsystem

$$x_{k+1} = (I_f - \varepsilon\widehat{L}_k)x_k - \varepsilon s\widehat{K}_k 1_l. \tag{2}$$

We define the disagreement $\xi_k \in \mathbb{R}^f$ as the difference between the followers' and the leaders' states, that is, $\xi_k = x_k - s1_f$. Substituting this definition into (2) and using the zero-row sum property of $[\widehat{L}_k\ \widehat{K}_k]$, the disagreement dynamics is

$$\xi_{k+1} = (I_f - \varepsilon\widehat{L}_k)\xi_k. \tag{3}$$

We say that (1) is mean square consentable if the disagreement dynamics in (3) is (asymptotically) mean square stable, that is, if $\lim_{k\to\infty} \mathbf{E}[\|\xi_k\|^2] = 0$ for any $\xi_0 \in \mathbb{R}^f$, see for example (Abaid & Porfiri, 2011). We comment that, although the focus of this paper is on mean square stability, equivalences with other second moment stabilities can be found, for example, in Feng, Loparo, Ji, and Chizeck (1992).

Following Zhou and Wang (2009), we use the asymptotic convergence factor of the error dynamics (3) to ascertain the mean square consentability of (1). This quantity is defined as

$$r_a = \sup_{\|\xi_0\|\neq 0} \lim_{k\to\infty} \left(\frac{\mathbf{E}\left[\|\xi_k\|^2\right]}{\|\xi_0\|^2}\right)^{1/k} \tag{4}$$

and it is less than one if and only if the system is mean square consentable, see for example (Abaid & Porfiri, 2011). The expected value of the disagreement norm can be written as $\mathbf{E}\left[\|\xi_k\|^2\right] = \mathrm{vec}(I_f)^{\mathrm{T}}\mathrm{vec}(\mathbf{E}[\xi_k\xi_k^{\mathrm{T}}])$. By iteratively applying equation (3), this