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Brief paper

Oscillation analysis of linearly coupled piecewise affine systems: Application to spatio-temporal neuron dynamics ☆

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ABSTRACT

This paper discusses oscillation analysis of (a large number of) linearly coupled piecewise affine (PWA) systems, motivated by various kinds of reaction–diffusion systems including cell-signaling dynamics and neural dynamics. We derive a sufficient condition under which the system shows an oscillatory behavior called Y-oscillation. It is known that the analysis of PWA systems is difficult due to their switching nature. An important feature of the result obtained is that, under the assumption that every subsystem has a specific property in common, the criteria can be rewritten in terms of coupling topology in an easily checkable way, so it is applicable to large scale systems. The results obtained are applied to theoretical investigation of the cardiac action potential generation/propagation represented by spatio-temporal FitzHugh–Nagumo equations.

1. Introduction

Many important oscillatory phenomena such as the circadian rhythms (Crosthwaite, Dunlap, & Loros, 1997; Goldbeter, 1995; Kholodenko, 2006) exist in the natural world. This has prompted much theoretical research on modeling and analysis of oscillatory phenomena, in particular for periodic orbits; e.g., monotonicity (Angeli & Sontag, 2008), the Poincaré–Bendixson theorem (Mallet-Paret, 1996; Mallet-Paret & Smith, 1990; Wang, Li, & Aihara, 2008), Hopf bifurcation (Mees & Chua, 1979; Stan, 2005; Stan & Sepulchre, 2007), and Poincaré/impact maps (Goncalves, Megretski, & Dahleh, 2003). Placed among these, the results in this paper are closely related to spatially distributed phenomena observed in reaction–diffusion systems (Turing, 1952). In Keener and Sneyd (1998), Kholodenko (2006) and Murray (2003), we can find many interesting biological examples with detailed analysis of partial differential equation (PDE) models.

In our work, we concentrate on large scale arrays consisting of mutually coupled nonlinear systems. This class covers spatially discretized forms of the PDE models mentioned above, as well as central pattern generator models (Collins & Stewart, 1993; Kopell & Ermentrout, 1988). For this class of coupled systems, various kinds of theoretical analyses of oscillatory phenomena have been obtained both in dynamical system theory (Heagy, Carrol, & Pecora, 1994; Hu, Yang, & Liu, 1998; Wu & Chua, 1995) and in the controls community (Angeli & Sontag, 2008; Jovanović, Arcak, & Sontag, 2008). These results often focus on networks with some specific coupling topology such as diffusive coupling and cyclic feedback.

Aside from the approaches listed above, in Pogromsky, Glad, and Nijmeijer (1999), Y-oscillatory behavior in diffusively coupled systems has been analyzed. This Y-oscillation, originally introduced by Tomberg and Yakubovich (1989), is a general notion of oscillatory phenomena that covers both periodic and aperiodic trajectories; see Definition 1 for the mathematical description. By introducing a new notion of semi-passivity, it has been proven in Pogromsky et al. (1999) and Pogromsky and Nijmeijer (2001) that there exists a diffusively coupled nonlinear system that is Y-oscillatory and whose identical subsystems are globally asymptotically stable at the origin. Furthermore, an approach similar to that of Pogromsky et al. (1999) and further based on Hopf/pitchfork bifurcation has been proposed in Stan (2005) and Stan and Sepulchre (2007) for global analysis of passive oscillators and their interconnection. This line of *passivity-based* approach provides a checkable condition for oscillatory phenomena including synchronization in large scale coupled nonlinear systems, when passivity (in a suitably weak sense) of all subsystems and the semi-positive definiteness of coupling matrices can be guaranteed. It should be noted that it is not necessarily easy to verify semi-passivity of general nonlinear systems though, in Steur, Tyukin, and Nijmeijer (2009), certain neuron models are shown to be semi-passive in a constructive manner.

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Concerning representation of nonlinearity, a piecewise affine (PWA) approach has been extensively adopted in mathematical physiology (Keener & Sneyd, 1998). In view of this, we derive an easily verifiable sufficient condition under which the linearly coupled PWA systems are Y-oscillatory. There are several numerical methods for the analysis of PWA systems, e.g., the pioneering paper (Johansson & Rantzer, 1998) and recent related works (Efimov & Fradkov, 2009; Goncalves et al., 2003; Salinas-Varela, Stan, & Goncalves, 2008). Unfortunately, it is still difficult to apply them to the systems considered here, since the state dimension and the number of modes quickly increases in the coupled dynamics. An important feature of the main result, which is obtained by utilizing the property of PWA systems explicitly, is that the criteria can be rewritten in terms of the connection topology when the underlying subsystem dynamics are identical. This provides a *scalable* criterion with respect to the number of subsystems.

The paper is organized as follows. In the next section, we describe how PWA approximations can represent important properties of the original nonlinear models. In Section 3, we formulate the problem, and then a sufficient condition for Y-oscillation is given. In Section 4, we focus on a more specific class of linearly coupled systems consisting of identical subsystems. We show that under a certain assumption they can be easily analyzed via eigenvalue decomposition of the coupling matrix. The results obtained are applied to theoretical investigation of the cardiac action potential generation/propagation represented by spatio-temporal FitzHugh–Nagumo equations. Section 5 concludes this paper.

Notation and conventions: For a matrix A, $\mathrm{eig}(A)$ denotes the set of all eigenvalues. A square matrix A is said to be Hurwitz if $\mathrm{eig}(A)$ is a subset of the open left complex half-plane. The $n \times n$ identity matrix is I_n. For a vector $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the Euclidean norm, that is, $\|x\| := \sqrt{x^{\mathrm{T}}x}$ where $^{\mathrm{T}}$ is the matrix transposition. The (block-)diagonal matrix and the Kronecker product are represented by diag and $\otimes$, respectively. For a set $\mathcal{S}$, $\mathrm{int}\,\mathcal{S}$, $\partial\mathcal{S}$ and $\bar{\mathcal{S}}$ denote the interior, boundary and closure of $\mathcal{S}$.

2. PWA approximation of dynamical models in mathematical physiology

In this section, we see how PWA systems can capture important properties of physiological models (FitzHugh, 1969; Hindmarsh & Rose, 1984; Hodgkin & Huxley, 1952; Nagumo, Arimoto, & Yoshizawa, 1962) through approximating the FitzHugh–Nagumo equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} g(X_1) - X_2 \\ \varepsilon(X_1 - bX_2) - \varepsilon\alpha \end{bmatrix}, \tag{1}$$

$$g(X_1) := -10X_1(X_1 - 1)(X_1 - 0.5) \tag{2}$$

where $\varepsilon, b > 0$ and α are real constants. We approximate the nonlinear term in (2) by the piecewise affine function depicted in Fig. 1:

$$g(X_1) \approx \tilde{g}(X_1) := \begin{cases} -5X_1 + 5, & \text{if } 0.8333 \le X_1, \\ -5X_1, & \text{if } X_1 \le 0.1667, \\ 2.5X_1 - 1.25, & \text{otherwise.} \end{cases}$$

We set $b = \varepsilon = 0.1$ and show the time response of $X_1(t)$ for $\alpha = 0.5$ and $\alpha = 0.2$ in Fig. 2. Though the initial state is the same ($[X_1(0), X_2(0)]^{\mathrm{T}} = [0.6, -0.1]^{\mathrm{T}}, [0.2, -0.1]^{\mathrm{T}}$), the behavior is completely different. In mathematical physiology, these properties are referred to as the *self-oscillation* and *excitability* (convergence after a possible temporal perturbation), and used to model the behavior of neural networks. See Section 3.3 for the actual roles of these properties in living organisms.

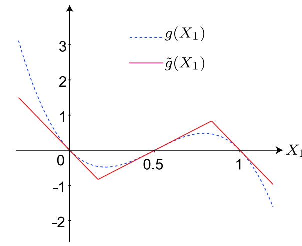

Fig. 1. Approximation of g.

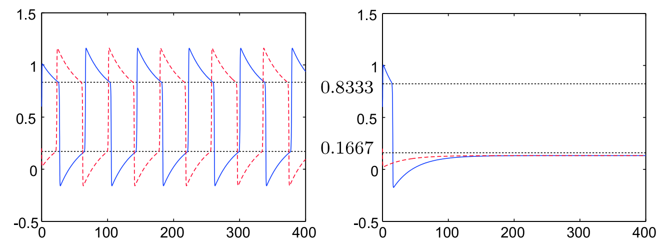

Fig. 2. The self-oscillatory property with $\alpha = 0.5$ (left) and the excitability property with $\alpha = 0.2$ (right).

In practice, we can increase the number of modes to obtain a sufficiently accurate PWA vector field, depending on the smoothness of the original nonlinearity and required accuracy. Note that it would sometimes be valid to approximate *physical systems* directly by PWA systems, in particular when the dynamics is discontinuous. Refer to Imura, Kashima, Kusano, Ikeda, and Morohoshi (2010) for examples and references related to these topics. However, PWA approximation may require more careful discussions for the case of complex behaviors such as bifurcation (Stan, 2005; Stan & Sepulchre, 2007). This topic is beyond the scope of this paper.

3. Linearly coupled PWA systems

In general, it is crucial to investigate not only an individual neural cell behavior, but also their large scale coupled array. In what follows we discuss the latter dynamics within the framework of linearly coupled PWA systems.

3.1. System representation

Let J be the number of subsystems. Throughout this paper, the index $j \in \mathbb{J} := \{1, 2, \ldots, J\}$ is used to express the jth subsystem. Next, let $\{\mathcal{S}_i\}_{i \in \mathbb{I}}$ be a family of closed subsets of $\mathbb{R}^{\bar{n}}$ indexed by mode labels $\mathbb{I} := \{1, 2, \ldots, L\}$. This plays the role of state partitions since we assume $\bigcup_{i \in \mathbb{I}} \mathcal{S}_i = \mathbb{R}^{\bar{n}}$ and $\mathcal{S}_i$ have disjoint interiors. Then, the system investigated in this paper is given by

$$\dot{x}_j = A_{i_j}^{(j)} x_j + b_{i_j}^{(j)} + D \sum_{k \in \mathbb{J}} \gamma_{jk} x_k, \quad \text{if } x_j \in \mathcal{S}_{i_j} \tag{3}$$

where $x_j \in \mathbb{R}^{\bar{n}}$ and $i_j \in \mathbb{I}$ denote the state variable and the mode of the jth subsystem, and $A_{i_j}^{(j)}, D \in \mathbb{R}^{\bar{n} \times \bar{n}}$, $b_{i_j}^{(j)} \in \mathbb{R}^{\bar{n} \times 1}$. When $D = 0$, every subsystem is an autonomous L-mode piecewise affine system with the switching signal i_j. In other words, the third term with nonzero D specifies the linear interaction between these PWA subsystems.

Note that the coupled system (3) is again a PWA system equipped with L^J-modes and the state variable

$$x(t) := \left[x_1(t)^{\mathrm{T}}, x_2(t)^{\mathrm{T}}, \ldots, x_J(t)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^n$$

with $n := J\bar{n}$. To see this, defining

$$\mathbf{i} := (i_1, i_2, \ldots, i_J) \in \mathbb{I}^J,$$