Automatica 50 (2014) 1349–1359

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Dynamic generalized controllability and observability functions with applications to model reduction and sensor deployment☆

Mario Sassano [a,1], Alessandro Astolfi [a,b]

[a] Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133, Rome, Italy

[b] Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK

ARTICLE INFO

Article history:
Received 23 September 2013
Received in revised form
28 January 2014
Accepted 4 February 2014
Available online 12 March 2014

Keywords:
Controllability and observability functions
Nonlinear systems
Model reduction
Sensor deployment

ABSTRACT

In this paper we introduce the notion of Dynamic Generalized Controllability and Observability functions for nonlinear systems. These functions are called *dynamic* and *generalized* since they make use of additional states (dynamic extension) and are such that partial differential inequalities are solved in place of equations. The presence of the dynamic extension permits the construction of classes of canonical controllability and observability functions without relying on the solution of any partial differential equation or inequality. The effectiveness of the proposed concept is validated by means of two applications: the model reduction problem via balancing and the sensor deployment problem in a continuous stirred tank reactor (CSTR).

1. Introduction

Minimal realization theory for linear systems provides the tool to determine a state-space description of minimal order, the impulse response of which matches the impulse response of the original system. In situations in which this approach cannot be pursued for practical or theoretical reasons, one may still be interested in finding a lower-order representation that *approximates* the behavior of the original model, thus solving the so-called model reduction problem (Antoulas, Sorensen, & Gugercin, 2001; Glover, 1984; Moore, 1981; Pernebo & Silverman, 1982; Scherpen, 1993; Scherpen & Van Der Schaft, 1994).

The first step towards the solution of the model reduction problem is the characterization of a *measure* of importance of the state components according to some desired criterion. Controllability and observability Gramians are well-known mathematical tools to describe linear systems in terms of their input/output behavior. In the case of locally asymptotically stable systems, this concept permits an *ordering* of the states with respect to their input/output energy, namely taking into account the control effort necessary to *steer* the system from a specific state to the origin in infinite time and the energy released by the output of the system initialized at a particular state, respectively.

In the linear framework, the controllability and observability functions are determined from the solutions of Lyapunov matrix equations and are related to the controllability and observability Gramians of the system. Mimicking the above ideas, controllability and observability functions have been defined also for general nonlinear systems (Scherpen, 1993). The controllability and observability functions of nonlinear systems are the solution of first-order partial differential equations, which may be hard or impossible to determine analytically. This aspect represents a serious drawback to the applicability of model reduction by balancing (of the controllability and the observability functions) to practical cases.

Once these functions have been determined, there exists a (nonlinear) change of coordinates such that, in the new coordinates, the functions are in the so-called *balanced* form (Scherpen, 1993). After the coordinates transformation the components of the state can be ordered according to the energy required to steer the state and the output energy released by the corresponding initial condition. Therefore, if a state demands a large amount of energy to be controlled, on one hand, and it is *hardly* observable (in terms of output energy), on the other hand, then clearly the contribution of the aforementioned state to the input/output behavior of the system is negligible and could be ignored in a lower-order approximation.

☆ This work is partially supported by the EPSRC Programme Grant Control For Energy and Sustainability EP/G066477 and by the MIUR under PRIN Project Advanced Methods for Feedback Control of Uncertain Nonlinear Systems. The material in this paper was partially presented at the 52st IEEE Conference on Decision and Control (CDC), December 10–13, 2012, Maui, Hawaii, USA. This paper was recommended for publication in revised form by Associate Editor Nathan Van De Wouw under the direction of Editor Andrew R. Teel.

E-mail addresses: mario.sassano@uniroma2.it (M. Sassano), a.astolfi@ic.ac.uk (A. Astolfi).

[1] Tel.: +39 0672597337; fax: +44 2075946282.

http://dx.doi.org/10.1016/j.automatica.2014.02.041
0005-1098/© 2014 Elsevier Ltd. All rights reserved.

In addition, in the case of unstable nonlinear systems several techniques have been proposed, such as *LQG*, *HJB* or H_∞ balancing. The latter has been introduced for linear systems in Mustafa and Glover (1991) and subsequently extended to the nonlinear case in Scherpen (1996). Moreover, it is shown in Scherpen and Van Der Schaft (1994) that the *HJB* singular value functions, obtained from the past and future energy functions, are strongly interconnected with the Graph Hankel singular value functions, derived by balancing the controllability and observability functions of the normalized coprime factorizations of the nonlinear system. As a result, the reduced order models obtained with the two different approaches coincide. Finally, the assumption of zero-state observability is relaxed in Gray and Mesko (1999) allowing for non-zero inputs in the definition of the observability function.

The main contribution of the paper consists in the definition of the notion of Dynamic Generalized Controllability and Observability functions. They are said to be *generalized* and *dynamic* since partial differential inequalities are solved, in place of equations, in an extended state-space, respectively. In fact, the additional state may be considered as a dynamic extension introducing auxiliary dynamics to combine with a positive definite function. While the notion of generalized Gramians has been introduced in the literature, see for instance (Prajna & Sandberg, 2005; Sandberg, 2010; Sandberg & Rantzer, 2004), the idea of considering *dynamic* Gramians is new. Interestingly, the main advantage of these functions over the classical controllability and observability functions is that the former can be constructively defined avoiding the explicit solution of any partial differential equation or inequality. Preliminary results about the notion of Dynamic Generalized Controllability and Observability function and its construction are reported in Sassano and Astolfi (2012b). Different from Sassano and Astolfi (2012b) an alternative notion of (matrix) algebraic $\bar{P}$ solution is considered here, which leads to significant simplifications in the construction of such functions and, moreover, in the application of the latter functions to the model reduction problem via optimal balancing. In addition, a detailed discussion about the *point-wise* minimization of the approximation error, intrinsically introduced by the proposed functions, and a description of the application of the latter functions to the problem of sensor deployment are reported.

The rest of the paper is organized as follows. Section 2 introduces the notion of Dynamic Generalized Controllability and Observability functions which can be constructed, as shown in Section 3, without relying on the solution of any partial differential equation or inequality. A class of Dynamic Generalized Controllability and Observability functions is constructively defined in Section 3. Finally, the effectiveness of the proposed notion is validated by means of two different examples in Section 4. To begin with, the application of the Dynamic Generalized Controllability and Observability functions to the problem of balancing and model reduction for nonlinear systems is discussed. Once the functions have been constructed, a nonlinear system can be transformed into a *dynamically balanced form* by means of a change of coordinates. Then, the optimal sensor deployment problem is discussed and a dynamic observability function is employed to determine whether it is more informative to measure the substrate or the biomass concentration in a continuous stirred tank reactor (CSTR). Conclusions are drawn and future work is outlined in Section 5.

2. Dynamic generalized controllability and observability functions

Consider a nonlinear system described by equations of the form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ the input, and $y(t) \in \mathbb{R}^p$ the output. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be sufficiently smooth and such that $f(0) = 0$ and $h(0) = 0$, without loss of generality. By the latter assumption, there exist, possibly not unique, continuous matrix-valued functions $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $H : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ such that $f(x) = F(x)x$ and $h(x) = H(x)x$, respectively, for all $x \in \mathbb{R}^n$. In what follows it is assumed that the linearization around the origin of system (1) is controllable and observable. In addition suppose that the system (1) is zero-state detectable, namely that $u(t) = 0$ and $y(t) = 0$, for all $t \geq 0$, imply that the state $x(t)$ tends to zero as time tends to infinity. Finally suppose that the zero equilibrium of the system (1) is locally asymptotically stable.

Definition 1 (*Scherpen, 1993*)**.** Consider the nonlinear system (1). The functions, if they exist,

$$L_c(\bar{x}) = \min_{u\in\mathcal{L}_2(-\infty,0),x(-\infty)=0,x(0)=\bar{x}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt \tag{2}$$

and

$$L_o(\bar{x}) = \frac{1}{2}\int_0^\infty \|y(t)\|^2 dt, \quad \text{s.t. } x(0) = \bar{x},\ u(t) \equiv 0, \tag{3}$$

are the controllability and observability functions, respectively, of system (1).

The function L_o may be unbounded if the zero equilibrium of system (1) is unstable, whereas the controllability function L_c is unbounded at $\bar{x}$ by convention if the state $\bar{x}$ cannot be reached from zero. For simplicity, we suppose throughout the paper that the controllability and observability functions are defined, namely take finite values, at least in a neighborhood of the origin. If the system (1) is linear, *i.e.*

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{4}$$

the controllability and observability functions are defined – with the assumption of asymptotic stability of the zero equilibrium point – as $L_c(\bar{x}) = \frac{1}{2}\bar{x}^\top \bar{P}^{-1}\bar{x}$ and $L_o(\bar{x}) = \frac{1}{2}\bar{x}^\top \bar{Q}\bar{x}$, respectively, where the matrices $\bar{P} = \bar{P}^\top > 0$ and $\bar{Q} = \bar{Q}^\top > 0$ are solutions of the Lyapunov equations $A\bar{P} + \bar{P}A^\top + BB^\top = 0$ and $A^\top\bar{Q} + \bar{Q}A + C^\top C = 0$, respectively. The notion of Dynamic Generalized Controllability and Observability functions is introduced in the two following definitions, respectively.

Definition 2. Consider the nonlinear system (1). A *Dynamic Generalized Controllability function* $\mathcal{V}_c$ is a pair $(\mathcal{D}_c, \mathcal{L}_c)$ defined as follows.

- $\mathcal{D}_c$ is the ordinary differential equation
$$\dot{\xi}_c = \phi_c(x, \xi_c), \tag{5}$$
with $\xi_c(t) \in \mathbb{R}^n$ and $\phi_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_c(0, 0) = 0$, smooth mapping.
- $\mathcal{L}_c : \Omega_c \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}_c(0, 0) = 0$ and $\mathcal{L}_c(x, \xi_c) > 0$ for all $(x, \xi_c) \in \Omega_c \setminus \{0\}$, is such that
$$\frac{\partial \mathcal{L}_c}{\partial x} f(x) + \frac{\partial \mathcal{L}_c}{\partial \xi_c}\phi_c(x, \xi_c) + \frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial x} g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \leq 0, \tag{6}$$
for all $(x, \xi_c) \in \Omega_c$. Moreover $(0,\ 0)$ is an asymptotically stable equilibrium point of the system
$$\dot{x} = -f(x) - g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top, \qquad \dot{\xi}_c = -\phi_c(x, \xi_c). \tag{7}$$