Automatica 50 (2014) 1444–1450

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Quadratic stability and stabilization of bimodal piecewise linear systems[☆]

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ARTICLE INFO

Article history:
Received 18 February 2013
Received in revised form
9 November 2013
Accepted 18 February 2014
Available online 28 March 2014

Keywords:
Piecewise linear systems
Bimodal systems
Common quadratic Lyapunov function
Quadratic stability
Feedback stabilization

ABSTRACT

This paper deals with quadratic stability and feedback stabilization problems for continuous bimodal piecewise linear systems. First, we provide necessary and sufficient conditions in terms of linear matrix inequalities for quadratic stability and stabilization of this class of systems. Later, these conditions are investigated from a geometric control point of view and a set of sufficient conditions (in terms of the zero dynamics of one of the two linear subsystems) for feedback stabilization are obtained.

1. Introduction

Common quadratic Lyapunov functions are among the most popular tools in the stability of linear switching systems, both for state-independent (Agrachev & Liberzon, 2001; Liberzon, 1999) and state-dependent switchings (Johansson & Rantzer, 1998). One of the main reasons behind their popularity is that (whenever exists) such Lyapunov functions can be efficiently computed via linear matrix inequalities. As such, providing sufficient conditions for stability in terms of feasibility of a set of linear matrix inequalities is highly popular in the literature of linear switching systems (Camlibel, Pang, & Shen, 2007; Pavlov, Pogromsky, Van De, & Nijmeijer, 2007). However, these conditions are rather computational in nature and often do not relate to the underlying structure of the system under study, in particular for the case of state-dependent switchings.

In this paper, we focus on a particular class of linear switching systems with state-dependent switchings, namely piecewise bimodal systems with a continuous vector field. In a way, these systems form the simplest class of piecewise affine systems. The main goal of the paper is to investigate the existence of a quadratic Lyapunov function for such systems with an eye towards the underlying geometric structure. It turns out that continuity of the underlying vector field leads to an alternative linear matrix inequality based condition for the existence of a common quadratic Lyapunov function. In turn this alternative condition enables us to look at the feedback stabilization problem from a geometric point of view. Indeed, one of the main results of the paper is to provide sufficient conditions for the existence of a stabilizing static state feedback for bimodal systems. These sufficient conditions are not of linear matrix inequality type but rather geometric conditions and involve the zero dynamics of one of the linear subsystems (and hence also the other due to continuity). We also compare the (open-loop) stabilizability conditions and those for the static state feedback stabilization.

The paper is organized as follows. In Section 2, we first introduce the class of bimodal systems as well as the quadratic stability notion under study. Then, we provide necessary and sufficient conditions for quadratic stability in terms of linear matrix inequalities. Section 3 deals with the feedback stabilization problem and provides necessary and sufficient conditions for the

[☆] The authors acknowledge The Scientific and Technological Research Council of Turkey (TUBITAK) for funding this research. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Constantino M. Lagoa under the direction of Editor Roberto Tempo.

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http://dx.doi.org/10.1016/j.automatica.2014.03.009
0005-1098/

existence of a static state feedback rendering the closed-loop system quadratically stable. After comparing the existing open-loop stabilizability conditions and those presented for the feedback stabilization, we provide a set of sufficient conditions for the feedback stabilization in terms of the zero dynamics of one of the linear subsystems. Finally, the paper closes with conclusions in Section 4 and Appendix which presents a technical lemma and its proof for the sake of completeness.

2. Quadratic stability of bimodal systems

Consider the bimodal piecewise affine system given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) + f + bu(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + f + bu(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and all vectors/matrices involved are of appropriate dimensions. Throughout this paper, we assume that the right-hand side is a continuous function in x, or equivalently, there exists a vector $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = ec^T. \tag{2}$$

In this case, the right-hand side of (3) is a Lipschitz continuous function. Hence, for each initial state x_0 and locally-integrable input u there exists a unique absolutely continuous function x such that (3) holds for almost all $t \in \mathbb{R}$ and $x(0) = x_0$.

Such bimodal systems can be encountered in a variety of applications sometimes artificially as approximations of nonlinear systems and sometimes naturally due to the intrinsic piecewise affine behaviour. Next, we illustrate an example for the latter case.

Example 1. As an example, consider the mechanical system shown in Fig. 1. We assume that all the elements are linear. Let x_1 and x_2 denote the displacements of the left and right cart from the tip of the leftmost spring, respectively. Also let the masses of the carts denoted by m_1 (for the left one) and m_2 (for the other), the spring constants by k' (for the leftmost one) and k (for the other), and the damping constant by d. Then, the governing differential equations can be given by

$$m_1\ddot{x}_1 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) - k' \max(-x_1, 0) = 0$$

$$m_2\ddot{x}_2 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = F$$

where F is the force that is applied to the right cart. By denoting the velocities of the left and right cars, respectively, by x_3 and x_4, one arrives at the following bimodal piecewise linear system

$$\dot{x} = \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-(k+k')}{m_1} & \dfrac{k}{m_1} & \dfrac{-d}{m_1} & \dfrac{d}{m_1} \\ \dfrac{-k}{m_2} & \dfrac{k}{m_2} & \dfrac{-d}{m_2} & \dfrac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \leqslant 0 \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-k}{m_1} & \dfrac{k}{m_1} & \dfrac{-d}{m_1} & \dfrac{d}{m_1} \\ \dfrac{-k}{m_2} & \dfrac{k}{m_2} & \dfrac{-d}{m_2} & \dfrac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \geqslant 0 \end{cases}$$

$$y = x_1$$

where $x = \mathrm{col}(x_1, x_2, x_3, x_4)$. Note that the condition (2) is satisfied for $e = \mathrm{col}(0, 0, -\frac{k'}{m_1}, 0)$.

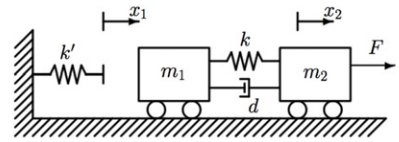

Fig. 1. Linear mechanical system with a one-sided spring.

More realistic applications of bimodal systems arising from one-sided springs can be found in for instance (Doris et al., 2008, Section 3); (Doris, van de Wouw, Heemels, & Nijmeijer, 2010, Section 4). These papers deal with observer design and disturbance attenuation problems, respectively, for a continuous bimodal system arising as a mathematical model of two steel beams, one supported at both ends by two leaf springs whereas the other (which is located parallel to the first one) clamped at both ends acting as a one-sided spring.

Other control systems applications in which bimodal systems arise intrinsically include for instance (van der Heijden, Serrarens, Camlibel, & Nijmeijer, 2007) where clutch engagement problem has been studied and Vanek, Bokor, Balas, and Arndt (2007).

In addition to engineering applications, continuous bimodal systems are also encountered in various other contexts. Examples from the area of dynamical systems include Carmona, Fernandez-Garcia, Fernandez-Sanchez, Garcia-Medina, and Teruel (2012), Carmona, Fernandez-Garcia, and Freire (2011), Michelson (1986) and Webster and Elgin (2003). In what follows, we illustrate a bimodal system arising in the study of certain partial differential equations.

Example 2. The so-called Michelson system was originally studied in Michelson (1986) in the context of the steady solutions of the Kuramoto–Sivashinsky (partial differential) equations and further studied in for instance (Carmona, Fernandez-Sanchez, & Teruel, 2008; Webster & Elgin, 2003). It can be given (after a suitable similarity transformation) as a bimodal system of the form (1) where

$$A_i = \begin{bmatrix} 0 & -1 & (-1)^i \lambda(1+\lambda^2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{for } i \in \{1, 2\},$$

$$f^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad c^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and $\lambda \in \mathbb{R}$ is a constant. Note that the continuity assumption (2) is satisfied with $e^T = \begin{bmatrix} -2\lambda(1+\lambda)^2 & 0 & 0 \end{bmatrix}$.

Next, we focus on particular cases of (1) where $f = b = 0$, that is continuous bimodal systems of the form:

$$\dot{x}(t) = \begin{cases} A_1 x(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) & \text{if } c^T x(t) \geqslant 0. \end{cases} \tag{3}$$

We say that the bimodal system (3) is *quadratically stable* if there exists a quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all $x \neq 0 \in \mathbb{R}^n$ and $\dot{V}(x(t)) < 0$ for all state trajectories x of (3) with $x(t) \not\equiv 0$. Equivalently, the system (3) is quadratically stable if and only if there exists a common quadratic Lyapunov function for the linear subsystems, that is there exists a symmetric positive definite matrix P such that

$$A_i^T P + PA_i < 0 \tag{4}$$

with $i \in \{1, 2\}$.

Note that quadratic stability of systems of the form (3) naturally yields (local) Lyapunov stability (see e.g. Khalil (2002)) of possibly non-zero equilibrium points of bimodal systems of the form:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + f & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + f & \text{if } c^T x(t) \geqslant 0. \end{cases} \tag{5}$$