Automatica 48 (2012) 191–197

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Set membership approximation of discontinuous nonlinear model predictive control laws[☆]

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ARTICLE INFO

Article history:
Received 14 July 2010
Received in revised form
30 March 2011
Accepted 30 June 2011
Available online 11 October 2011

Keywords:
Predictive control
Function approximation
Discontinuous control
Nonlinear systems
Set membership approximation theory

ABSTRACT

In this paper, the use of Set Membership (SM) methods is investigated, in order to derive off-line an approximation of a discontinuous nonlinear model predictive control (NMPC) law. The approximating function can then be evaluated on-line, instead of solving the nonlinear program embedded in the NMPC scheme. This way, a significant decrease of computational times may be obtained, thus allowing the application of NMPC also to systems with "fast" dynamics. It is shown that the knowledge of the discontinuities is needed to achieve an approximated controller with arbitrarily small approximation error. By exploiting such a knowledge, SM techniques already developed for the case of continuous NMPC laws are generalized in order to approximate discontinuous ones. The stability of the origin of the closed loop system with the approximated control law is analyzed, and a numerical example is employed to illustrate the features of the proposed approach.

1. Introduction

In nonlinear model predictive control (NMPC, see e.g. Mayne, Rawlings, Rao, and Scokaert (2000)) the control action is computed by means of a Receding Horizon (RH) strategy, which requires at each sampling time the solution of a nonlinear program (NLP, see e.g. Nocedal and Wright (2006)), where the systems state x (and, possibly, other measured parameters and reference variables) is a parameter in the optimization. For time invariant systems, the solution of the NLP defines a static nonlinear function $\kappa(x)$, denoted in this paper as the "nominal" control law. In the last decade, a significant research effort has been devoted to the problem of efficient implementation of NMPC laws, motivated by the objective of applying this control strategy also to systems with relatively "fast" dynamics, in which the employed sampling period does not allow the real-time solution of the NLP. A possible viable approach is the use of an approximated NMPC law $\hat{\kappa} \approx \kappa$, derived off-line and then evaluated on-line instead of solving the NLP. Contributions in the field of approximated NMPC can be found e.g. in Canale, Fagiano, and Milanese (2009, 2010), Canale, Fagiano, Milanese, and Novara (2010), Grancharova and Johansen (2009), Johansen (2004), Parisini and Zoppoli (1995), Raimondo et al. (2011), Summers, Raimondo, Jones, Lygeros, and Morari (2010) and Ulbig, Olaru, Dumur, and Boucher (2007), using various approaches. In particular, approximation techniques based on Set Membership (SM) theory have been developed and studied in Canale et al. (2009, 2010) and Canale, Fagiano, Milanese et al. (2010). In this framework, approximated NMPC laws with guaranteed accuracy (in terms of a bound on the pointwise error $\kappa(x) - \hat{\kappa}(x)$) and consequent performance and stability properties have been derived, with the assumption of continuity of κ over the compact subset $\mathcal{X}$ of the state space considered for the approximation. Although the assumption of continuity of κ holds for MPC with linear and "almost linear" models (Mayne & Michalska, 1990) and for a series of problems with nonlinear models and/or nonlinear constraints, it is well-known that NMPC laws may be discontinuous and that there exist systems that cannot be stabilized with continuous control laws (see e.g. Lazar, Heemels, Bemporad, and Weiland (2007), Michalska and Vinter (1994), Meadows, Henson, Eaton, and Rawlings (1995) and Messina, Tuna, and Teel (2005)). In these cases, the guaranteed properties of the existing SM approaches do not hold anymore. In the described context, the contributions of this paper are (a) to show through a motivating example that the knowledge of the

[☆] This research has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement no. PIOF-GA-2009-252284 — Marie Curie project "Innovative Control, Identification and Estimation Methodologies for Sustainable Energy Technologies". The material in this paper was partially presented at the joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, December 16–18, 2009, Shanghai, PR China. This paper was recommended for publication in revised form by Associate Editor Lalo Magni under the direction of Editor Frank Allgöwer.

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doi:10.1016/j.automatica.2011.09.039

discontinuities is needed in order to achieve an arbitrarily small pointwise approximation error, which is required to retain the closed-loop stability properties, (b) to use such a knowledge to derive an approximation of a discontinuous NMPC law, using an SM approach, and (c) to study the closed loop system stability when the SM approximated law is used. Finally, a numerical example is employed to show the features of the proposed approximation technique.

2. Problem settings

Consider the following nonlinear state space model:

$$x_{t+1} = f(x_t, u_t) + w_t, \quad \|w_t\|_2 \le \mu \tag{1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and the control input, respectively, and $w_t \in \mathbb{R}^n$ is an unknown but bounded disturbance.

Assumption 1. Function f in (1) is continuous with respect to u_t, i.e. for any fixed value $\bar{x}_t$ the function $\bar{f}(u_t) = f(\bar{x}_t, u_t)$ is continuous over $\mathbb{R}^m$.

The control objective is to regulate the system state to the origin under some input and state constraints, represented by a convex set $\mathbb{X} \subseteq \mathbb{R}^n$ and a compact set $\mathbb{U} \subseteq \mathbb{R}^m$, both containing the origin in their interiors, in which the state and input values x_t and u_t should be kept, respectively. In NMPC, the control law $u_t = \kappa(x_t)$ is defined implicitly by a RH strategy, in which at each sampling instant the following NLP has to be solved:

$$\min_U J(U, x_t) \doteq \sum_{k=0}^{N_p-1} L(x_{t+k|t}, u_{t+k|t}) + \Phi(x_{t+N_p|t}) \tag{2a}$$

s.t.

$$x_{t+k|t} \in \mathbb{X}, \quad k = 1, \ldots, N_p \tag{2b}$$

$$u_{t+k|t} \in \mathbb{U}, \quad k = 0, \ldots, N_p \tag{2c}$$

Stabilizing constraints (2d)

where $x_{t+k|t}$ denotes k steps ahead state prediction using the model (1), given the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$ and the "initial" state $x_{t|t} = x_t$, and $U = \left[u_{t|t}^T, \ldots, u_{t+N_c-1|t}^T\right]^T$ is the vector of the control moves to be optimized. $N_p \in \mathbb{N}$ and $N_c \in \mathbb{N}$, $N_c \le N_p - 1$ are the prediction and the control horizons, respectively. The remaining predicted control moves $[u_{t+N_c|t}, \ldots, u_{t+N_p-1|t}]$ can be computed with different strategies (Mayne et al., 2000). The optimal cost and its optimizer are indicated as $J^*(U^*(x_t), x_t)$ and $U^*(x_t)$. It is assumed that the optimization problem (2) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the "feasibility set", so that $\kappa : \mathcal{F} \to \mathbb{U}$. The application of the RH controller gives rise to the following nonlinear state feedback configuration:

$$x_{t+1} = f(x_t, \kappa(x_t)) + w_t = F^0(x_t) + w_t = F_w^0(x_t, w_t). \tag{3}$$

The system (3) will be also referred to as the "nominal" closed loop system in the following. The set of solutions of (3) at the generic time instant t, starting from the initial condition $x_0 \in \mathcal{F}$ and considering all the possible realizations of the disturbance sequences $\{w\}$, is indicated here as $\mathcal{S}_\mu^0(t, x_0) \doteq \{\phi_w^0(t, x_0) = \underbrace{F_w^0(F_w^0(\cdots F_w^0(x_0, w_0)))}_{t \text{ times}}, \forall \{w_k\} : \|w_k\|_2 \le \mu, k = 0, \ldots, t\}$. With a proper choice of the cost function J and of the stabilizing constraints (2d), it is possible to achieve robust closed loop stability and constraint satisfaction properties in the presence of the disturbance w. In particular, existing approaches for robust NMPC exploit constraint tightening, state contraction constraints, terminal set constraints, min–max formulations and input-to-state stability (ISS) techniques (see e.g. Chisci, Rossiter, and Zappa (2004), Goodwin, Seron, and De Dona (2005), Lazar, Heemels, Roset, Nijmeijer, and van den Bosch (2008), Mayne et al. (2000), Magni and Scattolini (2006) and Pin, Raimondo, Magni, and Parisini (2009)). In this paper, κ will be approximated over a compact set $\mathcal{X} \in \mathcal{F}$. An important theoretical issue to be addressed in the approximation of κ concerns the capability to provide a guaranteed approximation accuracy and the evaluation of the effects of the approximation on the closed loop stability properties. In Canale et al. (2009, 2010) and Canale, Fagiano, Milanese et al. (2010) it has been shown that, in the context of SM approximation theory, it is possible to derive an approximated control law κ^{SM}, based on the preliminary off-line computation of a finite number ν of nominal control moves, that enjoys the following three properties:

$$\kappa^{\text{SM}} : \mathcal{X} \to \mathbb{U} \tag{4a}$$

$$|\kappa(x) - \kappa^{\text{SM}}(x)| \le \zeta < \infty, \quad \forall x \in \mathcal{X} \tag{4b}$$

$$\lim_{\nu \to \infty} \zeta = 0. \tag{4c}$$

Properties (4a)–(4c), namely satisfaction of input constraints, bounded approximation accuracy with a finite bound ζ and convergence of ζ to zero, have been proved to be sufficient to be able to achieve closed loop stability and guaranteed regulation precision when the function κ^{SM} is employed for feedback control (Canale et al., 2009). However, the above-mentioned results rely on the assumption of continuity of κ in addition to its stabilizing properties, while it is known that, depending on the system model, the constraints and the chosen objective function, the nominal NMPC law may be discontinuous. In this case, in general, it is not possible to achieve the key property (4c), unless some other information on the discontinuities of κ is available. Moreover, the absence of property (4c) may lead to instability or to a reduction of the region of attraction for the origin of the closed-loop system. This concept will be illustrated in the next section, through a motivating example.

3. A motivating example

Consider the following system model:

$$x_{t+1} = a x_t + [b s(x_t) + c] u_t \tag{5}$$

where $x_t, u_t \in \mathbb{R}$, $s(x_t) = -1$ if $x_t < 1$ and $s(x_t) = 1$ if $x_t \ge 1$. Consider the NLP (2) with $U = u_{t|t}$, cost function $J(U, x_t) = x_{t+1|t}^2 + R u_t^2$, $R > 0$, and the stabilizing constraint $g(x_t, u_t) \le 0$, with $g(x_t, u_t) = x_{t+1|t}^2 - \alpha x_t^2$, $\alpha \in (0, 1)$ (contraction constraint). The contraction constraint ensures closed loop stability and it can be easily noted that a Lyapunov function for the closed loop system is $V(x) = x^2$. For any given value of x_t, the cost and constraint functions are convex (quadratic) functions of u_t. Thus, the Karush–Kuhn–Tucker (KKT) conditions (see e.g. Nocedal and Wright (2006)) are sufficient for global optimality. Assuming that $a = 5, b = 1, c = 0.5, R = 2, \alpha = 0.81$, by imposing the KKT conditions the following explicit optimal control law is obtained:

$$\kappa(x_t) = -\frac{4.1}{s(x_t) + 0.5} x_t. \tag{6}$$

Function κ (6) is clearly discontinuous at $x_t = 1$, however let us assume that this information is not available for the approximation. Assume now that the approximation of κ has to be carried out on the set $\mathcal{X} = [-0.5, 1.5]$, and consider the following $\nu = 8$ nominal control values as part of the prior information on $\kappa : \tilde{u} = \{\kappa(\tilde{x}) : \tilde{x} \in \mathcal{X}_\nu\}$, where $\mathcal{X}_\nu = \{-0.50, -0.24, 0.02, 0.28, 0.54, 0.80, 1.06, 1.32\}$. Moreover, consider the Nearest Point (NP) approximation (see e.g. Canale et al. (2009)) of function (6), i.e. $\kappa^{\text{NP}}(x) = \kappa(\tilde{x}^{\text{NP}})$, where $\tilde{x}^{\text{NP}} = \arg\min_{\tilde{x} \in \mathcal{X}_\nu} \|x - \tilde{x}\|_2$. The nominal