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Brief paper

Optimal synergetic control for fractional-order systems☆

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ABSTRACT

Nowadays, the control of fractional-order system is one of the most popular topics in control theory. Recent studies have demonstrated the interest of fractional calculus both for systems modeling in many areas of science and engineering and for robust controller design. Thus, several research contributions have been devoted to the extension of control theory to fractional-order systems. Synergetic control was introduced in power electronics and other industrial processes. The benefit of this control scheme has been recognized for both integer-order linear and nonlinear systems. In this paper, a fractional-order synergetic control for fractional-order systems is proposed. Both linear and nonlinear cases are considered. The macro-variable is defined by the fractional-order integral of state variables. Optimality and stability properties are analyzed. A numerical example is investigated to confirm the effectiveness of the proposed method.

1. Introduction

During the last two decades, considerable work has demonstrated the interest of fractional-order differentiation and integration operators in many areas of science and engineering. It has been shown that the fractional differentiation operator permits one to model, with greater accuracy, some physical systems which exhibit hereditary, diffusion, and viscoelasticity properties. Fractional dynamics can be encountered in various systems such as viscoelastic materials, electrochemical processes, dielectric polarization, thermal systems, transmission and acoustic, chaos and fractals, electrical machines, and many others. Several books (Kilbas, Srivastava, & Trujillo, 2006; Monje, Chen, Vinagre, Xue, & Feliu, 2010; Podlubny, 1999a; Sabatier, Agrawal, & Machado, 2007) provide a good source of references on fractional calculus and its applications.

Fractional-order integration and differentiation operators have also received great attention in control theory. The main use of these fractional-order operators lies in the design of fractional-order controllers which enhance the robustness and the performance of the controlled system. For linear systems, in the frequency domain, a fractional-order Tilde Integral Derivative (TID) controller (Lune, 1994) and the well-known CRONE (Commande Robuste Ordre Non Entier) controllers (Sabatier, Oustaloup, Garcia Iturricha, & Lanusse, 2002) were the first to be developed. Later, the fractional-order PID controller (Podlubny, 1999b) and the fractional-order robust PID controller (Monje et al., 2005) were proposed. Other well-known control strategies designed for integer-order systems have been extended to fractional-order systems. State-space fractional design methods based on pole placement were developed in Farges, Moze, and Sabatier (2010).

The synergetic control scheme developed by Kolesnikov (2000) has been successfully applied in many industrial applications, mainly in the area of power systems (Kondratiev, Dougal, Santi, & Veselov, 2004; Santi, Monti, Proddutur, & Dougal, 2003). Synergetic control is a very attractive control strategy for nonlinear systems. The objective of the control is to force the system to operate on a manifold defined by a macro-variable. This macro-variable is selected according to the control specification. When the system trajectories reach the manifold, the system dynamics motion is governed by a linear first-order differential equation (Jiang & Dougal, 2004). Synergetic control shares with sliding mode control (Utkin, 1992) the same objective to force the closed-loop system to move on a desired manifold. Synergetic control operates at low frequencies and it does not have chattering (Santi et al., 2003). The reaching law is obtained to assure hitting the sliding surface asymptotically. High-order sliding mode (Fridman & Levant, 1996) can also prevent chattering by approaching the sliding manifold asymptotically, as reported in Shtessel, Shkolnikov, and Brown (2003). In Nusawardhama, Zak, and Crossley (2007), the authors have shown that the synergetic control strategy for integer-order

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systems can be derived from the calculus of variations and that it possesses optimality properties. In addition, with such synergetic control, the speed of convergence to the sliding manifold can be controlled. This is not the case for sliding mode control.

Structure variable controllers, including sliding mode control techniques, are also considered within the context of fractional-order differentiation operators. First, sliding mode control has been applied to linear fractional-order systems (Calderon, Vinagre, & Feliu, 2006; Efe, 2008; Si-Ammour, Djennoune, & Bettayeb, 2009). However, limited work has been reported for the sliding mode control of fractional nonlinear systems (Dadras & Momeni, 2010; Tavazoei & Haeri, 2007). These contributions consider the first-order sliding mode. Second-order sliding mode control has been considered only for linear fractional-order systems (Pisano, Rapaić, Jeličić, & Usai, 2010). On the other hand, in spite of the work already done (Agrawal, 2004; Li & Chen, 2008), the optimal control problems of fractional-order systems need to be investigated.

The objective of this paper is to develop a fractional-order optimal synergetic control for fractional-order systems. Both nonlinear and linear systems are considered. The contribution in this paper is twofold. First, the proposed synergetic control law is an attractive alternative for the control of nonlinear fractional-order systems. Second, it offers a new solution to the optimal control of linear and nonlinear fractional-order systems.

Starting from a fractional integral manifold as introduced in Si-Ammour et al. (2009), we show that the fractional-order synergetic control is an optimal control law which minimizes a linear quadratic performance index. Even if the system is governed by fractional-order nonlinear differential equations, the desired dynamic of the macro-variable on the manifold is governed by a first-order differential equation. The optimal closed-form solution of the control law is easily obtained for both linear and nonlinear fractional-order systems, compared to the solution proposed in Li and Chen (2008) for the linear quadratic regulator (LQR) problem. Finally, the stability of the closed-loop system is established.

The rest of the paper is organized as follows. In Section 2, definitions of the fractional-order integral and derivative are recalled. In Section 3, the optimal synergetic control for nonlinear fractional-order systems is developed. The stability of the closed-loop system is analyzed. Section 4 is devoted to linear fractional-order systems. The synergetic fractional optimal control is constructed and the stability of the closed-loop system is also established. In Section 5, a numerical example is considered to test the efficiency of the developed method.

2. Fractional differentiation and integration

Let $L_1[a\, b]$ denote the space of Lebesgue-integrable real-valued functions $f(t)$ of the variable t, which represents the time, on the interval $[a,\ b]$, $a, b \in \Re_+$, such that $0 \leq a < b < \infty$, and let $\Re_+$ denote the non-negative real numbers set. Let $AC[a\, b]$ be the space of functions $f(t)$ which are continuous on $[a\, b]$, and we denote by AC^k the space of real-valued functions $f(t)$ which have continuous derivatives up to order $k-1$ such that $f^{(k-1)}(t) \in AC[a\, b]$, where $f^{(i)}(t)$ is the ith integer-order derivative of $f(t)$.

Definition 1 (*Kilbas et al., 2006*). Let $f(t) \in L_1[a\, b]$ be a function of the variable t, $t \in [a,\ b]$. The fractional integral of order $\alpha \in \Re_+$ is defined by the Riemann–Liouville integral

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{1}$$

where the Euler Gamma function $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty \nu^{\alpha-1} e^{-\nu} d\nu. \tag{2}$$

Remark 1. Formula (1) is called the left-side fractional integral. The right-side fractional integral has also been defined (Kilbas et al., 2006). Integral (1) represents the convolution of the function $f(t)$ with the kernel function $\phi(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

Three definitions of fractional-order derivatives have been introduced in the literature, namely the Riemann–Liouville derivative, the Caputo derivative and the Grünwald–Letkinov derivative (Kilbas et al., 2006; Podlubny, 1999a). The Riemann–Liouville fractional derivative is used in this paper. This derivative takes into account more precisely the infinite-dimensional characteristic of the system.

Let $\alpha \in \Re_+$ be the fractional order of the derivative such that $s-1 < \alpha < s$; s denotes any integer and a denotes the initial time.

Definition 2 (*Kilbas et al., 2006*). The Riemann–Liouville fractional derivative of order α of $f(t) \in AC^s[a\, b]$; $t \in [a\, b]$ is defined as

$$ {}_a^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(s-\alpha)} \frac{d^s}{dt^s} \int_a^t (t-\tau)^{s-\alpha-1} f(\tau) d\tau. \tag{3}$$

The initial time is taken to be zero. For simplicity, we use the following notation:

$$I_0^\alpha f(t) \triangleq I^\alpha f(t)$$
$$ {}_0^{RL}D_t^\alpha f(t) \triangleq D^\alpha f(t)$$
$$Df(t) \triangleq \frac{df(t)}{dt} \triangleq \dot{f}(t).$$

Some useful properties of the Riemann–Liouville fractional integral and derivative are summarized below (Kilbas et al., 2006).

Proposition 1 (*Kilbas et al., 2006*). *Let $f(t)$ be a real-valued function of the variable $t \in [a\, b] \subset \Re_+$. Then the following hold.*

(1) $I^0 f(t) = f(t)$ *and* $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$, $f(t) \in L_1[a\, b]$, $\forall \alpha, \beta \in \Re_+$.
(2) $D^\alpha I^\beta f(t) = D^{\alpha-\beta} f(t)$, $\forall \alpha, \beta \in \Re_+$.
(3) $D^\alpha(c_1 f_1(t) + c_2 f_2(t)) = c_1 D^\alpha f_1(t) + c_2 D^\alpha f_2(t)$.

3. Optimal synergetic control for nonlinear fractional-order systems

A fractional-order time-invariant nonlinear system is described by a state-space representation which involves fractional derivatives of state variables $x_i(t)$, $i = 1, 2, \ldots, n$. Consider the class of the nonlinear system

$$D^{[\alpha]} x(t) = f(x(t)) + g(x(t))u(t), \tag{4}$$

where $x(t) = [x_1\, x_2 \ldots x_n]^T \in \Re^n$ denotes the n-dimensional state vector, $u(t) = [u_1\, u_2 \ldots u_m]^T \in \Re^m$ is the control input vector, and $D^{[\alpha]} = [D^{\alpha_1}\, D^{\alpha_2} \ldots D^{\alpha_n}]^T$ is the fractional differentiation vector operator of orders $\alpha_i \in \Re_+$, $i = 1, 2, \ldots, n$. If all orders α_i are equal, that is $\alpha_i = \alpha$, $i = 1, 2, \ldots, n$, then the system is called a commensurate fractional-order system (Matignon, 1998). Consider the case of the commensurate fractional-order system described by

$$D^\alpha x(t) = f(x(t)) + g(x(t))u(t), \tag{5}$$

where $D^\alpha x(t) = [D^\alpha x_1\, D^\alpha x_2 \ldots D^\alpha x_n]^T$, $f(x(t))$ and $g(x(t))$ are smooth vector fields for $x(t) \in \mathcal{D} \subset \Re^n$, where $\mathcal{D}$is a compact set containing the origin. α is the order of the derivative such that $0 < \alpha < 1$. We take the initial time $t_0 = 0$. We assume that the system was at rest at time a, $-\infty < a < 0$ and that the system was initialized in the past, beginning at time a. As mentioned in Hartley and Lorenzo (2002) and Sabatier, Merveillant, Malti, and Oustaloup (2010), a fractional-order system requires complicated