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Brief paper

Observers and initial state recovering for a class of hyperbolic systems via Lyapunov method ☆

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ABSTRACT

Recently the problem of estimating the initial state of some linear infinite-dimensional systems from measurements on a finite interval was solved by using the sequence of forward and backward observers Ramdani, Tucsnak, and Weiss (2010). In the present paper, we introduce a direct Lyapunov approach to the problem and extend the results to the class of semilinear systems governed by wave and beam equations with boundary measurements from a finite interval. We first design forward observers and derive Linear Matrix Inequalities (LMIs) for the exponential stability of the estimation errors. Further we obtain simple finite-dimensional conditions in terms of LMIs for an upper bound T^* on the minimal time, that guarantees the convergence of the sequence of forward and backward observers on $[0, T^*]$ for the initial state recovering. This T^* represents also an upper bound on the observability time. For observation times bigger than T^*, these LMIs give upper bounds on the convergence rate of the iterative algorithm in the norm defined by the Lyapunov functions. In our approach, T^* is found as the minimal dwelling time for the switched exponentially stable (forward and backward estimation error) systems with the different Lyapunov functions (Liberzon, 2003). The efficiency of the results is illustrated by numerical examples.

1. Introduction

Estimation of the initial state of a distributed parameter system from its input and output functions measured over some finite time interval is an important problem in engineering, oceanography, meteorology and medical imaging (see e.g. Ramdani et al., 2010, and the references therein). For the linear exactly observable distributed parameter system, the initial state can be recovered from the measured segment of the input and output functions by inverting the Gramian operator of the system (see, for instance Tucsnak & Weiss, 2009, Section 6.1), and this may be numerically very challenging. However, this is not applicable to nonlinear systems.

Recently the problem of estimating the initial state of some infinite-dimensional systems from measurements on a finite interval has been solved by using a sequence of forward and backward observers (Auroux & Nodet, 2012; Ramdani et al., 2010). For finite-dimensional systems this idea has appeared in Auroux and Blum (2005). In Ramdani et al. (2010) the condition on the convergence of the iterative procedure is given in terms of the bounds on the norms of the semigroups generated by the operators of the forward and backward estimation error equations. It is not easy to find the latter bounds. Moreover, the results of Ramdani et al. (2010) (and the convergence results of Auroux & Nodet, 2012) are confined to the linear time-invariant case.

It is of interest to develop consistent methods that are capable of utilizing nonlinear distributed parameter models and of providing simple conditions for the convergence of forward and backward observers. The LMI approach (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) is definitely among such methods. For time-delay systems, this approach allowed to solve various control problems in terms of simple finite-dimensional conditions (see e.g. Fridman & Shaked, 2002; Gu, Kharitonov, & Chen, 2003; Richard, 2003, and the references therein). Its extension to distributed parameter systems has been started in Fridman and Orlov (2009a,b).

The LMI approach to observers and initial state recovering of distributed parameter systems is the primary concern of the present paper, where we consider semilinear 1-d wave and beam equations. We start with the design of forward observers and derive LMIs for the exponential stability of the estimation errors. Though the stability of the beam equation has been studied in the literature via direct Lyapunov method (see e.g. Guo & Yang, 2009; Krstic, Guo, Balogh, & Smyshlyaev, 2008), these are the first LMIs for the exponential stability. Their derivation is based on Wirtinger's inequality (Hardy, Littlewood, & Polya, 1934) and on the application of the S-procedure (Yakubovich, 1977).

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Further we find LMIs that give an upper bound T^* on the minimal time, that guarantees the convergence of the sequence of forward and backward observers on $[0, T^*]$ for the recovery of the initial state. This T^* represents also an upper bound on the exact observability time. The continuous dependence of the reconstructed initial state on the measurements follows from the integral input-to-state stability of the corresponding error system (see Angeli, Sontag, & Wang, 2000), which is guaranteed by the LMIs for the exponential stability. For observation times larger than T^*, these LMIs give upper bounds on the convergence rate of the iterative algorithm in the norm defined by the Lyapunov functions. Finding T^* is similar to finding the minimal dwelling time for the switched exponentially stable systems with different Lyapunov functions (Liberzon, 2003). It appears that the LMIs are not conservative for the linear homogeneous wave equation recovering the analytical value of the minimal observability time. Some preliminary results for wave equations were presented in Fridman (2013).

1.1. Notation and preliminaries

Throughout the paper $\mathbb{R}^n$ denotes the n dimensional Euclidean space with the norm $|\cdot|$, the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. Functions, continuous (continuously differentiable) in all arguments, are referred to as of class C (of class C^1). $L^2(0, 1)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [0, 1]$ with the corresponding norm $\|z\|_{L^2} = \sqrt{\int_0^1 z^2(\xi)d\xi}$. $\mathscr{H}^1(0, 1)$ is the Sobolev space of absolutely continuous scalar functions $z : [0, 1] \to R$ with $\frac{dz}{d\xi} \in L^2(0, 1)$. $\mathscr{H}^2(0, 1)$ is the Sobolev space of scalar functions $z : [0, 1] \to R$ with absolutely continuous $\frac{dz}{d\xi}$ and with $\frac{d^2z}{d\xi^2} \in L^2(0, 1)$.

The following inequalities will be useful:

Lemma 1.1. *Let $z \in \mathscr{H}^1(0, 1)$ be a scalar function with $z(0) = 0$ or $z(1) = 0$. Then Wirtinger's inequality holds (Hardy et al., 1934)*

$$\int_0^1 z^2(x)dx \le \frac{4}{\pi^2}\int_0^1 z_x^2(x)dx. \tag{1.1}$$

Moreover,

$$\max_{x\in[0,1]} z^2(x) \le \int_0^1 z_x^2(x)dx. \tag{1.2}$$

2. Observers and initial state recovering: wave equation

2.1. Observers for semilinear wave equations

Consider the following one-dimensional semilinear wave equation

$$z_{tt}(x, t) = \frac{\partial}{\partial x}[a(x)z_x(x, t)] + f(z_x(x, t), x, t),$$
$$t \ge t_0,\ x \in (0, 1), \tag{2.1}$$

under the boundary conditions

$$z(0, t) = 0, \qquad z_x(1, t) = 0. \tag{2.2}$$

Here subscripts denote the corresponding partial derivatives, f is a C^2 function with uniformly bounded first partial derivatives in the two first variables.

The initial conditions are given by

$$z(x, t_0) = z_1(x), \qquad z_1(0) = 0, \qquad z_{1x}(1) = 0,$$
$$z_t(x, t_0) = z_2(x). \tag{2.3}$$

The smooth function $a(x)$ satisfies the following inequalities:

$$0 < a(1) \le a(x), \qquad a_x(x) \le 0, \quad \forall x \in (0, 1). \tag{2.4}$$

Let $g_1 > 0$ be the known bound on the derivative of $f(\xi, x, t)$ with respect to the first argument:

$$|f_\xi(\xi, x, t)| \le g_1 \quad \forall(\xi, x, t) \in \mathbb{R}^3. \tag{2.5}$$

The boundary measurements are given by $y(t) = z_t(1, t)$, $t \ge t_0$.

The boundary-value problem (2.1), (2.2) can be represented as an abstract differential equation by defining the state $\zeta(t) = [\zeta_1(t)\ \zeta_2(t)]^T = [z(t)\ z_t(t)]^T$ and the operators

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ \frac{\partial}{\partial x}\left[a(x)\frac{\partial}{\partial x}\right] & 0 \end{bmatrix}, \qquad F(\zeta, t) = \begin{bmatrix} 0 \\ F_1(\zeta_1, t)\end{bmatrix},$$

where $F_1 : \mathscr{H}^1 \times R \to L^2(0, 1)$ is defined as $F_1(\zeta_1, t) = f(\zeta_{1x}(x), x, t)$ so that it is continuous in t for each $\zeta_1 \in \mathscr{H}^1$. The differential equation is

$$\dot\zeta(t) = \mathscr{A}\zeta(t) + F(\zeta(t), t), \quad t \ge t_0 \tag{2.6}$$

in the Hilbert space $\mathscr{H} = \mathscr{H}_L^1(0, 1) \times L^2(0, 1)$, where

$$\mathscr{H}_L^1(0, 1) = \left\{\zeta_1 \in \mathscr{H}^1(0, 1) | \zeta_1(0) = 0\right\}$$

and $\|\zeta\|^2_{\mathscr{H}} = \|\zeta_{1x}\|^2_{L^2} + \|\zeta_2\|^2_{L^2}$. The operator $\mathscr{A}$ with the dense domain

$$\mathscr{D}(\mathscr{A}) = \Big\{(\zeta_1, \zeta_2)^T \in \mathscr{H}^2(0, 1) \bigcap \mathscr{H}_L^1(0, 1) \times \mathscr{H}_L^1(0, 1) \Big| \zeta_{1x}(1) = 0\Big\}$$

is m-dissipative and hence it generates a strongly continuous contraction semigroup $\mathbb{T}$ (Pazy, 1983). Due to (2.5) the following Lipschitz condition holds:

$$\|F_1(\zeta_1, t) - F_1(\bar\zeta_1, t)\|_{L^2} \le g_1\|\zeta_{1x} - \bar\zeta_{1x}\|_{L^2} \tag{2.7}$$

where $\zeta_1, \bar\zeta_1 \in \mathscr{H}_L^1(0, 1)$, $t \in \mathbb{R}$. Then by Theorem 6.1.2 of Pazy (1983), a unique continuous mild solution $\zeta(\cdot)$ of (2.6) in $\mathscr{H}$ initialized by

$$\zeta_1(t_0) = z_1 \in \mathscr{H}_L^1(0, 1), \qquad \zeta_2(t_0) = z_2 \in L^2(0, 1), \tag{2.8}$$

i.e. a unique solution of the integral equation

$$\zeta(t) = \mathbb{T}(t - t_0)\zeta(t_0) + \int_{t_0}^t \mathbb{T}(t - s)F(\zeta(s), s)ds \tag{2.9}$$

exists in $C([t_0, \infty), \mathscr{H})$. Moreover, this solution is locally Lipschitz in the initial state (i.e. for all $T > 0$ the mapping $(z_1, z_2) \to \zeta$ is Lipschitz from $\mathscr{H}$ to $C([t_0, T], \mathscr{H})$). Note that $F : \mathscr{H} \times [t_0, \infty) \to \mathscr{H}$ is continuously differentiable. If $\zeta(t_0) \in \mathscr{D}(\mathscr{A})$, then this mild solution is in $C^1([t_0, \infty), \mathscr{H})$ and it is a classical solution of (2.1), (2.2) with $\zeta(t) \in \mathscr{D}(\mathscr{A})$ (see Theorem 6.1.5 of Pazy, 1983).

We suggest a nonlinear Luenberger type observer of the form

$$\hat z_{tt}(x, t) = \frac{\partial}{\partial x}[a(x)\hat z_x(x, t)] + f(\hat z_x(x, t), x, t),$$
$$t \ge t_0,\ x \in (0, 1) \tag{2.10}$$

under the boundary conditions

$$\hat z(0,\ t) = 0, \qquad \hat z_x(1, t) = k[y(t) - \hat z_t(1, t)], \tag{2.11}$$

and the initial conditions $[\hat z(\cdot, t_0), \hat z_t(\cdot, t_0)]^T \in \mathscr{H}$, where $k > 0$ is the injection gain. The well-posedness of (2.10), (2.11) will be