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Brief paper

Discrete-time controllability for feedback quantum dynamics[☆]

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ABSTRACT

Controllability properties for discrete-time, Markovian quantum dynamics are investigated. We find that, while in general the controlled system is not finite-time controllable, feedback control allows for arbitrary asymptotic state-to-state transitions. Under further assumptions on the form of the measurement, we show that finite-time controllability can be achieved in a time that is twice the dimension of the system, and we provide an iterative procedure to design the unitary control actions.

1. Introduction

For any controlled system, an in-depth study of its controllability properties under the available control capabilities is the necessary premise to the design of effective controls addressing some given task. For quantum systems, in particular, controllability properties have been studied mostly considering continuous-time models in the presence of open-loop, coherent controls (Albertini & D'Alessandro, 2003; Altafini, 2002, 2003, 2004; D'Alessandro, 2007; Schirmer, Fu, & Solomon, 2001). In this setting, the evolution is deterministic and the problem can be studied with the tools of geometric control theory. Indeed, for classical deterministic systems it makes little sense to distinguish open-loop and feedback controllability: the fact that the control law can benefit from partial or complete information on the system trajectory does not modify the reachable set from a given initial state. In the quantum case, however, the introduction of measurements alone modifies the dynamical model by introducing a stochastic behavior, which has to be carefully taken into account. Considering the "open-loop" effect of measurements is not enough: the ability of *conditioning the control choice on the measurement outcomes changes significantly the controllability properties*. Continuous-time controllability of open-loop quantum dynamical semigroups have been studied in Altafini (2003, 2004) and Dirr, Helmke, Kurniawan, and Schulte-Herbrüggen (2009). Some preliminary ideas about discrete-time, open-system controllability have been also previously explored in Lloyd and Viola (2001), with a focus on engineering open dynamics, (Doherty, Jacobs, & Jungman, 2001), where feedback with strong measurements is shown to be enough for pure-state preparation, and (Wu, Pechen, Brif, & Rabitz, 2007), the main focus being on the existence of dynamics connecting any given pair of states. In this paper we investigate the controllability properties of controlled, Markovian *discrete-time quantum dynamics* closed loop. Open-loop controllability is a generic property for closed quantum systems, and this motivates our assumption of unitary controllability of the discrete-time systems we consider. On the other hand, by introducing generalized measurements and closing the loop with conditional control actions, the dynamics drastically changes and our main results shall focus on this setting. We will first present three simple examples illustrating how feedback may affect controllability and its limitations. Next, we will prove that, under generic conditions on the chosen measurement, feedback allows for *asymptotic state-to-state controllability*. Lastly, we will study a particular, yet not so restrictive in practice, class of controlled dynamics that exhibit *finite-time feedback state-to-state controllability*. As a byproduct of the proof of finite-time controllability, an explicit way to construct the sequence of control actions is provided. Remarkably, the (maximum) number of feedback steps needed to obtain any desired state-to-state transition is twice the size of the system's Hilbert space.

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2. Discrete-time quantum dynamics and controllability notions

In this paper we will consider finite-dimensional quantum systems. Let us introduce some basic notation: to the quantum system of interest is associated an Hilbert space $\mathcal{H} \sim \mathbb{C}^N$. Let $\mathfrak{M}(\mathcal{H})$ denote the set of linear operators on $\mathcal{H}$, and $\dagger$ the adjoint of operators (and consistently the transpose-conjugate for their matrix representations). Self-adjoint (Hermitian) operators are denoted by $X = X^\dagger \in \mathfrak{H}(\mathcal{H})$, and are associated to *observable* variables for the system, while the set of unitary operators is denoted by $\mathfrak{U}(\mathcal{H})$. In a quantum statistical framework, a *state* for the system is associated to trace-one, self-adjoint and positive-semidefinite operator ρ. Let us denote by $\mathfrak{D}(\mathcal{H})$ the set of states or *density operators*. The subset $\mathfrak{P}(\mathcal{H}) \subset \mathfrak{D}(\mathcal{H})$ denotes the set of rank-one orthogonal projectors, the *pure states*. $\mathfrak{D}(\mathcal{H})$ is a convex set, whose extreme points are the pure states $\mathfrak{P}(\mathcal{H})$, and its border $\delta\mathfrak{D}(\mathcal{H})$ contains all the states that are not full rank.

We shall consider *generalized measurements*, with a finite number of possible outcomes labeled by an index k. Assume a system is in the state ρ: a generalized measurement is associated to a decomposition of the identity $\sum_k M_k^\dagger M_k = I$, $M_k \in \mathfrak{M}(\mathcal{H})$ that allows to compute the probability of measuring the k-th outcome as

$$\mathbb{P}_\rho(k) = \mathrm{tr}(M_k \rho M_k^\dagger),$$

and the conditioned state after the measurement as

$$\rho|_k = \frac{1}{\mathbb{P}_\rho(k)} M_k \rho(t) M_k^\dagger. \quad (1)$$

A particular case is represented by direct measurements of observables, or *projective* measurements: consider an observable $X \in \mathfrak{H}(\mathcal{H})$, with spectral representation $X = \sum_k x_k \Pi_k$, $\sum_k \Pi_k = \sum_k \Pi_k^2 = I$. The eigenvalues correspond to the possible outcomes of the measurement, labeled by k, and the probabilities and conditioned states can be computed by the formulas above with $M_k = \Pi_k$. In the open quantum system setting, general physically admissible evolutions are described by linear, Completely Positive and Trace Preserving (CPTP) maps (Nielsen & Chuang, 2002). Any CPTP map $\mathcal{E}$ via the Kraus–Stinespring theorem (Kraus, 1983) admits explicit representations of the form

$$\mathcal{E}[\rho] = \sum_k M_k \rho M_k^\dagger \quad (2)$$

also known as Operator-Sum Representation (OSR) of $\mathcal{E}$, where ρ is a density operator and $\{M_k\}$ a family of operators such that the completeness relation

$$\sum_k M_k^\dagger M_k = I \quad (3)$$

is satisfied. We refer the reader to e.g. Alicki and Lendi (1987) and Nielsen and Chuang (2002) for a detailed discussion of the properties of quantum operations and the physical meaning of the complete-positivity property. Clearly, given (1), the expectation of the state after a measurement with unknown outcome leads to a map of the form (2). In the following we will deal with controlled open quantum models evolving in discrete time on the set of density operators. The dynamics will be generically described by

$$\rho(t+1) = \mathcal{E}(\rho(t), \vec{u}(t)), \quad (4)$$

with $\rho(\cdot) \in \mathfrak{D}(\mathcal{H})$ and $\vec{u}(t) \in \mathcal{U}$, and where $\mathcal{U}$ is the set of controls. Let us agree that $\mathfrak{R}_T(\rho)$ denotes the reachable set from ρ in T steps. We now introduce some relevant definitions of controllability properties. The system (4) is said to be

Pure state to Pure state Controllable (PPC) in T steps: if for every $\rho_0 = |\psi\rangle\langle\psi| \in \mathfrak{P}(\mathcal{H})$, $\mathfrak{R}_T(\rho_0) \supseteq \mathfrak{P}(\mathcal{H})$.

Density operator to Density operator Controllable (DDC) in T steps: if for every $\rho_0 \in \mathfrak{D}(\mathcal{H})$, $\mathfrak{R}_T(\rho_0) = \mathfrak{D}(\mathcal{H})$.

Analogous definition can be given for *Pure state to Density operator Controllable (PDC)* and *Density operator to Pure state Controllable (DPC)*. Clearly, being $\mathfrak{P}(\mathcal{H}) \subset \mathfrak{D}(\mathcal{H})$, it holds that DDC $\Longrightarrow$ DPC $\Longrightarrow$ PPC, and DDC $\Longrightarrow$ PDC $\Longrightarrow$ PPC. Weaker, *asymptotic* versions of the same controllability properties are of particular interest when dealing with discrete-time systems coming from sampling continuous time models: the definition follows from the ones given above by substituting the reachable set with

$$\bar{\mathfrak{R}}(\rho) = \mathrm{clo}\left(\bigcup_{T=0}^{\infty} \mathfrak{R}_T(\rho_0)\right),$$

with clo(X) denoting the closure of the set X. In terms of the *dynamical propagator*, we say that a system is

Unitary controllable (UC) in T steps: if for any $U \in \mathfrak{U}(\mathcal{H})$ there exist a choice of T controls $\vec{u}_i$ such that, $\mathcal{E}_i(A) = \mathcal{E}(A, \vec{u}_i)$,

$$U\rho U^\dagger = \mathcal{E}_T \circ \cdots \circ \mathcal{E}_1(\rho), \quad \forall \rho \in \mathfrak{D}(\mathcal{H}).$$

Kraus map controllable (KC) in T steps: if given any CPTP $\mathcal{E}$ there exists a choice of T controls such that $\mathcal{E} = \mathcal{E}_T \circ \cdots \circ \mathcal{E}_1$.

Some immediate relationships between the notions and the state controllability ones are KC $\Longrightarrow$ DDC, and KC $\Longrightarrow$ UC $\Longrightarrow$ PPC. The first implication has also been highlighted in Wu et al. (2007). It can be easily derived considering a *constant* mapping from $\mathfrak{D}(\mathcal{H})$ to $\rho_f \in \mathfrak{D}(\mathcal{H})$, ρ_f being the target state. This map can be extended to a linear CPTP map on $\mathfrak{M}(\mathcal{H})$, and hence it admits an OSR (by Kraus–Stinespring theorem Kraus, 1983; Nielsen & Chuang, 2002).

3. Feedback controllability

3.1. Discrete-time feedback control

We introduce here a discrete-time, Markovian feedback control scheme (Belavkin, 1983; Doherty et al., 2001; James, 2004; Lloyd & Viola, 2001), that has been recently studied in depth in Bolognani and Ticozzi (2010) focusing on stabilization problems. Assume that we can

(i) Enact a *fixed*, given generalized measurement associated to an OSR $\{M_k\}$;
(ii) Engineer a set of arbitrary unitary control action $U_k(t) \in \mathfrak{U}(\mathcal{H})$ at each time t, choosing U_k when the k-th outcome of the measurement is obtained.

While (ii) may appear a strong assumption, it is very reasonable for discrete-time systems that emerge from *sampling* of continuous time evolutions. In fact, it is well known that, in the absence of measurements and external noise, the (continuous time) Schrödinger equation for the propagator is generically controllable even in the presence of a single control Hamiltonian (D'Alessandro, 2007). In Sontag (1982, 1984), Sontag proved that if a dynamical system on a simply connected group is controllable (in continuous time), then it is sampled controllable (and asymptotically sampled controllable), namely there exist a sufficiently fast sampling frequency and a piecewise-constant control choice that realizes the desired transition. In our setting, this means that in a regime of *good control* (Shabani & Jacobs, 2008), any unitary operator can be generated in a finite number of steps by sufficiently fast sampled control.

Under this assumptions, if the state at time t was $\rho(t)$, the state at time $t+1$ conditioned to the k-th outcome of the generalized measurement is

$$\rho(t+1)|_k = \frac{U_k(t) M_k \rho(t) M_k^\dagger U_k^\dagger(t)}{\mathrm{tr}(M_k^\dagger M_k \rho(t))}.$$