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Brief paper

Optimal algorithms for trading large positions☆

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ABSTRACT

In this paper, we are concerned with the problem of efficiently trading a large position on the market place. If the execution of a large order is not dealt with appropriately this will certainly break the price equilibrium and result in large losses. Thus, we consider a trading strategy that breaks the order into small pieces and execute them over a predetermined period of time so as to minimize the overall execution shortfall while matching or exceeding major execution benchmarks such as the volume-weighted average price (VWAP). The underlying problem is formulated as a discrete-time stochastic optimal control problem with resource constraints. The value function and optimal trading strategies are derived in closed form. Numerical simulations with market data are reported to illustrate the pertinence of these results.

1. Introduction

With the increasing number of institutional investors like mutual funds, pension funds, hedge funds and private equity firms, who usually trade large positions on the market place, the automatic execution of those orders with computer programs have been growing exponentially over the last two decades. Algorithmic trading has improved over time; a wide variety of algorithms are actually being used to trade large positions with various levels of efficiency. Major financial institutions nowadays maintain one proprietary trading unit which uses algorithmic trading to execute large orders. One of the main issues that arise when trading large positions is the impact that the execution has on the share price itself. It becomes evident that the first thing that one should consider is to minimize the implementation shortfall of the trade, and then one can consider how the execution has fared with main market benchmarks. However, despite the wide range of algorithms used in electronic trading, the problem of effectively measuring the performance of these algorithms in light of various market parameters is still open. In fact there is no formal consensus among practitioners on which benchmark to use in order to measure the performance of a given algorithm. Among the various benchmarks available, the volume-weighted average price (VWAP) and the time-weighted average price (TWAP) are considered as particularly reliable quality measures for a large execution order.

Various aspects of this problem have been studied in the literature, both from the academic viewpoint as well as from that of industry. Among many other studies we may cite the work of Almgren and Chriss (1999, 2000). These authors had mainly used the mean–variance optimization to study the minimization of the implementation shortfall in the execution of large orders. They have proposed a wide variety of trading strategies under the assumption that the share price follows an arithmetic Brownian motion process. Moreover, selling rule problems have also generated great interests in the literature; one can refer to the work of Pemy and Zhang (2006), who have proposed optimal selling strategies under the regime switching framework. In addition, Helmes (2004) considered computational issues of the selling rule by using a linear programming approach. Pemy, Yin, and Zhang (2008) studied the liquidation of a large block of stock under regime switching within the framework of stochastic optimal control with state constraints.

In this paper, we study this problem as a discrete-time optimal control problem. In fact we propose two main trading algorithms. One mainly focuses on minimizing the execution shortfall and the other focuses on maximizing the trading VWAP. The share price follows a model based on the discrete version of the geometric Brownian motion in both of our trading algorithms; this is in clear contrast to the work of others, where the share price follows an arithmetic Brownian motion. The problem of minimizing the implementation shortfall is framed as a linear–quadratic tracking control problem with resource constraints. In both cases the value

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functions and trading strategies are obtained in closed form. Moreover the strategies proposed in this paper are mainly selling strategies; in subsequent work, we will extend these methods and combine selling and buying activities in both sell-side and buy-side algorithms.

The rest of this paper is organized as follows. In Section 2, we propose an algorithm minimizing execution shortfall. In Section 3, an optimal VWAP algorithm is obtained. In Section 4, we present numerical implementations of our algorithms using real market data.

2. An algorithm minimizing execution shortfall

Consider an investor who holds a large number of shares, say M, of a given stock and wants to sell his/her entire position by the time T. We assume that the share price of the stock follows the dynamics

$$x_{k+1} = g(x_k, m_k, \xi_k) = x_k + x_k\mu\tau + \sigma x_k\xi_k\sqrt{\tau} + f(m_k),\quad x_0 = x, \tag{1}$$

where x_k and m_k respectively represent the share price and the number of shares sold by the investor. The quantity τ is the time unit; it can roughly be taken as $\tau = T/N$ where N is the total number of times that shares are traded up to the time T. The ξ_k, $k = 1, 2, \ldots$, are independent and normally distributed random variables with mean 0 and variance 1. The volatility of the stock is σ and its return is μ. The function f represents the relative impact on the share price of trading a certain number of shares. More precisely, if we frequently put large blocks of shares in the market, this will definitely bring the price down. In order to model that effect, we have added a linear function of the number of shares traded, m_k, to our price dynamics. Thus, $f(x) = \lambda x$, $\lambda \in (-1, 0)$. In Almgren and Chriss (1999, 2000) the stock price follows an additive Brownian motion model. In this work, we propose instead a model based on the more traditional geometric Brownian motion.

Definition 2.1. 1. A selling strategy or policy is a finite sequence of nonnegative numbers $\pi = (m_k)_k$ such that $\sum_{k=0}^N m_k = M$.

2. Given a selling strategy π, the volume-weighted average price of π is

$$VWAP(\pi) = \frac{\sum m_k x_k}{\sum m_k}. \tag{2}$$

3. The market volume-weighted average price (VWAP) is defined as

$$VWAP_m = \frac{\sum v_k x_k}{\sum v_k}, \tag{3}$$

where v_k is the volume of shares traded at the kth selling period.

We denote by $\bar{x}$ the arrival price of the selling execution order, in other words, the actual share price when the execution order is given to the trader. The expected execution shortfall S is then obtained from the formula

$$S = \sum_{k=0}^N \mathbf{E} m_k x_k - M\bar{x},$$

where $\mathbf{E}$ represents the probability expectation.

Roughly speaking, our goal is to find a selling strategy that will in some way minimize the expected execution shortfall. For that we will use the optimal control framework; we will treat this problem as an optimal tracking control. In other words, we will be looking for a strategy that will enable us to sell our asset at prices x_k which are as close as possible to the arrival price $\bar{x}$. We define the tracking error $e_k = x_k - \bar{x}$; our goal is to drive this error as close as possible to zero. It is clear that the tracking error process $(e_k)_k$ follows the dynamics

$$e_{k+1} = e_k + e_k\mu\tau + e_k\xi_k\sigma\sqrt{\tau} + \lambda m_k + \bar{x}\mu\tau + \bar{x}\sigma\xi_k\sqrt{\tau},\quad \text{with } e_0 = x - \bar{x}. \tag{4}$$

We define the function h as follows:

$$h(e, \xi, m) = e + e\mu\tau + e\xi\sigma\sqrt{\tau} + \lambda m + \bar{x}\mu\tau + \bar{x}\sigma\xi\sqrt{\tau}.$$

Thus our controlled dynamics is then

$$e_{k+1} = h(e_k, \xi_k, m_k) \quad \text{with } e_0 = x_0 - \bar{x}.$$

The execution shortfall can be given as a function of our tracking error; thus

$$S = \sum_{k=0}^N \mathbf{E} m_k x_k - M\bar{x} = \sum_{k=0}^N \mathbf{E} m_k e_k. \tag{5}$$

Given that we want to find selling strategies that minimize $S = \sum_{k=0}^N \mathbf{E} m_k e_k$ over all possible selling strategies, we will thus formulate this problem as an optimal control problem. We will use the linear–quadratic regulator to drag the error process $(e_k)_k$ as close as possible to zero as we are selling. This will certainly help us to reduce the execution shortfall; moreover with the LQR control, we will easily derive the control process and the value function.

2.1. The linear–quadratic regulator

In this section we will be looking for a control strategy that substantially reduces our tracking error e_k as we trade; consequently this strategy will also in some way reduce our execution shortfall. We will be using a linear–quadratic regulator to control the tracking error e_k. We consider the following control problem:

$$(\mathscr{P}) \begin{cases} \text{minimize: } \sum_{k=0}^N \mathbf{E}(m_k^2 + e_k^2) \\ \text{with: } e_{k+1} = e_k + e_k\mu\tau + e_k\xi_k\sigma\sqrt{\tau} + \lambda m_k \\ \qquad\qquad + \bar{x}\mu\tau + \bar{x}\sigma\xi_k\sqrt{\tau}, \\ \quad e_0 = x_0 - \bar{x}, \\ \text{subject to: } \sum_{k=0}^N m_k = M, \quad \text{and} \\ \qquad m_k \geq 0,\ k = 0, \ldots, N. \end{cases} \tag{6}$$

This optimal control problem has a global resource constraint $\sum_{k=0}^N m_k = M$. We define our performance index as follows:

$$J(e; (m_k)) = \sum_{k=0}^N \mathbf{E}(e_k)^2 + m_k^2. \tag{7}$$

In order to take into account the constraints $\sum_{k=0}^N m_k = M$ and $m_k \geq 0$ for all k, we add a Lagrange multiplier $\gamma \geq 0$ to the performance index J. For any initial error e, and a tracking control policy $(m_k)_{k\leq N}$, we set

$$\tilde{J}(e, \gamma; (m_k)) = \sum_{k=0}^N \mathbf{E}(e_k)^2 + \gamma m_k + m_k^2. \tag{8}$$

The N-stage value function of our tracking control problem is defined as follows:

$$V_N(e, \gamma) = \min_{(m_k)_k} \tilde{J}(e, \gamma; (m_k)_k) = \min_{(m_k)_k} \sum_{k=0}^N \mathbf{E}(e_k)^2 + \gamma m_k + m_k^2. \tag{9}$$

We will solve this control problem for a wide range of values γ, and for suitable values of M, we can adjust γ such that the control variable $(m_k(\gamma))_k$ satisfies the condition $\sum_{k=0}^N m_k(\gamma) = M$.