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Tube-based robust sampled-data MPC for linear continuous-time systems[☆]

Marcello Farina [1], Riccardo Scattolini

Dipartimento di Elettronica e Informazione, Politecnico di Milano, Milan, Italy

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ABSTRACT

This note extends to the continuous-time case the "tube-based" approach for the design of discrete-time robust model predictive control (MPC) algorithms developed in Mayne, Seron, and Raković (2005). This extension is of interest in view of the simplicity and popularity of the method as well as of the industrial relevance of continuous-time implementations of MPC. The proposed robust control law is composed of two terms: (1) a sampled-data MPC control law and (2) a continuous-time state feedback term.

1. Introduction

The wide popularity of model predictive control (MPC) in both academia and industry has motivated the development of robust MPC algorithms, capable of dealing with model uncertainties, parameter variations, and external disturbances; see for example Limon et al. (2009) and references therein. Among the many solutions proposed, the "tube-based" approach for linear systems described in Mayne et al. (2005) is one of the most elegant and potentially applicable, since it implies on-line computations comparable with those required by standard (nominal) MPC algorithms. However, most of the robust MPC algorithms available nowadays (including Mayne et al., 2005) are based on a discrete-time formulation of the plant under control, while in industrial practice it is often preferred to resort to a continuous-time description of the system and to use a MPC that operates with piecewise constant control signals; see e.g. Chen and Allgöwer (1998) and Magni and Scattolini (2004). Notable exceptions of robust MPC algorithms for continuous-time systems are reported in Raff, Sinz, and Allgöwer (2008) and Rubagotti, Raimondo, Ferrara, and Magni (2011).

For these reasons, in this paper the robust MPC algorithm developed in Mayne et al. (2005) is extended to linear continuous-time systems affected by an unknown, but bounded disturbance. The main contribution consists in maintaining the simplicity of Mayne et al. (2005) and, at the same time, allowing for the use of a continuous-time plant model. To achieve this twofold objective, a robust MPC control law made up of two terms is proposed: the first one is a piecewise constant term computed as the solution of an MPC problem for the nominal system, while the second one is a continuous-time linear feedback law fed by the difference between the true and nominal state trajectories.

Notation. A matrix is Schur if all its eigenvalues lie in the interior of the unit circle, while it is Hurwitz if all its eigenvalues have negative real part. The symbol $\oplus$ denotes the Minkowski sum (Mayne et al., 2005); $\mathcal{B}_\delta(0) \subset \mathbb{R}^n$ is a ball of radius δ. The distance of a point z from a set $\mathbb{X}$ is $\mathrm{dist}(z, \mathbb{X}) := \inf\{\|z-x\| | x \in \mathbb{X}\}$; $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ are the maximum and the minimum eigenvalues of a matrix, respectively.

2. Preliminaries

Consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \tag{1}$$

where $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ are the state and input variables, respectively, and $w(t) \in \mathbb{W} \subseteq \mathbb{R}^n$ is a bounded unknown disturbance. The sets $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$ are convex and contain the origin. From (1), the nominal system can be obtained by neglecting the disturbance $w(t)$, that is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t) \tag{2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{u}(t) \in \mathbb{R}^m$ are the nominal state and input variables, respectively.

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E-mail addresses: farina@elet.polimi.it (M. Farina), scattolini@elet.polimi.it (R. Scattolini).

[1] Tel.: +39 02 2399 3599; fax: +39 02 2399 3412.

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The robust control law to be derived for system (1) is assumed to take the form

$$u(t) = \hat{u}(t) + K_c(x(t) - \hat{x}(t)) \tag{3}$$

where K_c is a feedback gain such that $A_c^{cl} = A + BK_c$ is Hurwitz. As for the term $\hat{u}$ in (3), it is assumed to be a piecewise constant signal. More specifically, letting $\{t_k\}, k \in \mathbb{I}_{\geq 0}$, be the set of sampling instants such that $t_{k+1} - t_k = T$ for all k, the aim of this paper is to propose a robust MPC scheme providing a sampled nominal control law $\hat{u}(t) = \hat{u}_k$ for all $t \in [t_k, t_{k+1})$ such that the closed-loop system (1)–(3) enjoys convergence and stability properties. For simplicity, in the following the notation $\hat{x}_k = \hat{x}(t_k)$ for all $k \in \mathbb{I}_{\geq 0}$ is also used. From (1)–(3), letting $z(t) = x(t) - \hat{x}(t)$ it follows that

$$\dot{z}(t) = A_c^{cl} z(t) + w(t) \tag{4}$$

and, since A_c^{cl} is Hurwitz, it is possible to define the robust positively invariant (RPI) set $\mathbb{Z}$ for (4) (see e.g. Rakovic & Kouramas, 2007) such that, if $z(t_0) \in \mathbb{Z}$ and $w(t) \in \mathbb{W}$ for all $t \geq t_0$, then $z(t) \in \mathbb{Z}$ for all $t \geq t_0$. In what follows, it is assumed that $\mathbb{Z} \subset \mathbb{X}$ and $K_c\mathbb{Z} \subset \mathbb{U}$, and that $\hat{\mathbb{X}} = \mathbb{X} \ominus \mathbb{Z}$ and $\hat{\mathbb{U}} = \mathbb{U} \ominus K_c\mathbb{Z}$ are neighborhoods of the origin. As in Mayne et al. (2005), it is assumed that the non-empty sets $\hat{\mathbb{X}}$ and $\hat{\mathbb{U}}$ exist. Note that they depend on the feedback gain K_c which is, in this respect, a design parameter to be selected properly.

An auxiliary sampled control law for the nominal system, together with its properties, is first derived. To this end, note that, given $\hat{x}_k = \hat{x}(t_k)$ and the sampled control variable $\hat{u}_k$, the continuous-time inter-sampling solution of (2) is

$$\hat{x}(t) = A_{zoh}(t - t_k)\hat{x}_k + B_{zoh}(t - t_k)\hat{u}_k, \quad t \in [t_k, t_{k+1}) \tag{5a}$$

where $A_{zoh}(\tau) = e^{A\tau}$ and $B_{zoh}(\tau) = \int_0^\tau e^{A(\tau-\eta)}Bd\eta$. Moreover, the discrete-time system representing (2) in the sampling instants is described by

$$\hat{x}_{k+1} = A_d\hat{x}_k + B_d\hat{u}_k \tag{5b}$$

where $A_d = A_{zoh}(T)$, $B_d = B_{zoh}(T)$.

Now assume that we know a sampled (auxiliary) control law $\hat{u}_k = K_d\hat{x}_k$ for the nominal system (2), such that $A_d^{cl} = A_d + B_dK_d$ is Schur. Given a symmetric matrix $\tilde{Q} > 0$ and constants $\gamma > 0$, $\gamma_2 > 0$ where $\gamma < \lambda_m(\tilde{Q})$, let the symmetric matrix Π be the unique positive definite solution of the following Lyapunov equation:

$$(A_d^{cl})^T \Pi A_d^{cl} - \Pi + \bar{Q} = 0$$

where $\bar{Q} = \int_0^T (A_{zoh}^{cl}(\eta))^T \tilde{Q} A_{zoh}^{cl}(\eta)d\eta + \gamma_2 I$ and $A_{zoh}^{cl}(\eta) = A_{zoh}(\eta) + B_{zoh}(\eta)K_d$. Then, from Lemma 1 in Magni and Scattolini (2004), there exist constants $T \in [0, +\infty)$ and $c > 0$ such that the set

$$\hat{\mathbb{X}}_f(K_d, T) = \{\hat{x} \mid \|\hat{x}\|_\Pi^2 \leq c\}$$

satisfies, for all $\hat{x}_k \in \hat{\mathbb{X}}_f$ and for all $t \in [t_k, t_{k+1})$,

$$\hat{x}(t) \in \hat{\mathbb{X}}, \qquad K_d\hat{x}_k \in \hat{\mathbb{U}} \tag{6a}$$

$$\|\hat{x}_{k+1}\|_\Pi^2 - \|\hat{x}_k\|_\Pi^2 \leq -\gamma \int_{t_k}^{t_{k+1}} \|\hat{x}(\eta)\|^2 d\eta - \gamma_2\|\hat{x}_k\|^2 \tag{6b}$$

where $\hat{x}(t)$ and $\hat{x}_{k+1}$ are computed as in (5a) and (5b), respectively, with $\hat{u}_k = K_d\hat{x}_k$.

A practical computational procedure consists in fixing the sampling period T according to well assessed criteria in digital control (Åström & Wittenmark, 1984). Then, a stabilizing gain K_d can be obtained with any synthesis technique and the matrix Π can be computed. Finally, the constant c can be derived by fixing the region $\hat{\mathbb{X}}_f$ where (6a) and (6b) are fulfilled.

3. The robust MPC problem formulation

Define the sequence $\hat{u}_{[k:k+N-1]} = \{\hat{u}_k, \ldots, \hat{u}_{k+N-1}\}$ of sampled inputs. Given $\hat{x}_k$ as the initial condition for (2) at time t_k, the nominal MPC problem $\hat{\mathcal{P}}_N(\hat{x}_k)$ can be stated as

$$\hat{J}_N^*(\hat{x}_k) = \min_{\hat{u}_{[k:k+N-1]}} J_N(\hat{x}_k, \hat{u}_{[k:k+N-1]}) \tag{7}$$

subject to system (2) and

$$\hat{u}_h \in \hat{\mathbb{U}}, \quad h = k, \ldots, k+N-1 \tag{8a}$$

$$\hat{x}(t) \in \hat{\mathbb{X}}, \quad t \in [t_k, t_k + NT) \tag{8b}$$

$$\hat{x}_{k+N} \in \hat{\mathbb{X}}_f. \tag{8c}$$

The cost function J_N is

$$J_N(\hat{x}_k, \hat{u}_{[k:k+N-1]}) = \int_{t_k}^{t_k+NT} (\|\hat{x}(t)\|_Q^2 + \|\hat{u}(t)\|_R^2)dt + \|\hat{x}(t_k + NT)\|_\Pi^2 \tag{9}$$

$$= \sum_{h=k}^{k+N-1} l(\hat{x}_h, \hat{u}_h) + \|\hat{x}_{k+N}\|_\Pi^2 \tag{10}$$

where $Q = Q' > 0$ and $R = R' > 0$ are such that

$$\lambda_M(Q) < \gamma, \qquad T\|K_d\|^2\lambda_M(R) < \gamma_2 \tag{11}$$

and γ, γ_2 are chosen as defined above.

Recalling that $\hat{u}(t)$ is constant over each sampling interval, the "stage cost" $l(\hat{x}_h, \hat{u}_h)$ in (10) is given by

$$l(\hat{x}_h, \hat{u}_h) = \int_{t_h}^{t_{h+1}} \|\hat{x}(\eta)\|_Q^2 d\eta + T\|\hat{u}_h\|_R^2. \tag{12}$$

Let $\hat{\mathbb{X}}_N$ denote the set of states $\hat{x}_k$ such that the problem $\hat{\mathcal{P}}_N(\hat{x}_k)$ admits a solution.

In $\hat{\mathcal{P}}_N(\hat{x}_k)$ the evolution of $\hat{x}_k$ is independent of $x(t)$ (i.e., a feasible choice is to let $\hat{x}_k$ evolve according to (2) where $\hat{u}_k$ is given by the solution to (7)), so that the resulting nominal control input is independent of the real state evolution. To avoid this, the algorithm developed in Mayne et al. (2005) implies that also $\hat{x}_k$ is a decision variable, while an additional constraint must guarantee that $\hat{x}_k$ lies in the neighborhood $\mathbb{Z}$ of the measured state $x(t)$. Therefore, the problem $\hat{\mathcal{P}}_N(\hat{x}_k)$ is modified accordingly and the problem $\mathcal{P}(x(t_k))$ to be solved at time t_k is

$$J_N^*(x(t_k)) = \min_{\hat{x}_k, \hat{u}_{[k:k+N-1]}} J_N(\hat{x}_k, \hat{u}_{[k:k+N-1]}) \tag{13}$$

subject to system (2), constraints (8) and the additional one

$$x(t_k) - \hat{x}_k \in \mathbb{Z}. \tag{14}$$

The optimal solution to $\mathcal{P}(x(t_k))$ is the pair $(\hat{x}_{k/k}, \hat{u}_{[k:k+N-1]/k})$, and $\hat{x}(t/t_k)$, $t \in [t_k, t_k + NT]$, is the solution to (5) obtained with $\hat{x}_{k/k}$ as initial condition and $\hat{u}_{[k:k+N-1]/k}$ as input sequence; furthermore $\hat{x}_{h/k} = \hat{x}(t_h/t_k)$ for $h = k, \ldots, k+N$.

For notational simplicity, define $\hat{u}_{k+N/k} = K_d\hat{x}_{k+N/k}$ and, for $t \in [t_k + NT, t_k + (N+1)T]$, $\hat{x}(t/t_k) = A_{zoh}^{cl}(t - (t_k + NT))\,\hat{x}_{k+N/k}$.

We denote with $\mathbb{X}_N$ the set of states $x(t_k)$ such that $\mathcal{P}_N(x(t_k))$ admits a solution.

According to (3), the control law for the perturbed system (1), for $t \in [t_k, t_{k+1})$, is given by

$$u(t) = \hat{u}_{k/k} + K_c(x(t) - \hat{x}(t/t_k)). \tag{15}$$

As in Mayne et al. (2005), the on-line computational burden related to the solution of $\mathcal{P}_N(x(t_k))$ is only slightly heavier than the one required by standard MPC. In fact, it is just necessary to enlarge