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Lie bracket approximation of extremum seeking systems☆

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ABSTRACT

Extremum seeking feedback is a powerful method to steer a dynamical system to an extremum of a partially or completely unknown map. It often requires advanced system-theoretic tools to understand the qualitative behavior of extremum seeking systems. In this paper, a novel interpretation of extremum seeking is introduced. We show that the trajectories of an extremum seeking system can be approximated by the trajectories of a system which involves certain Lie brackets of the vector fields of the extremum seeking system. It turns out that the Lie bracket system directly reveals the optimizing behavior of the extremum seeking system. Furthermore, we establish a theoretical foundation and prove that uniform asymptotic stability of the Lie bracket system implies practical uniform asymptotic stability of the corresponding extremum seeking system. We use the established results in order to prove local and semi-global practical uniform asymptotic stability of the extrema of a certain map for multi-agent extremum seeking systems.

1. Introduction

In diverse engineering applications one faces the problem of finding an extremum of a map without knowing its explicit analytic expression. Suppose, for example, one vehicle tries to minimize the distance to another vehicle. The only information available is the distance to the other vehicle. Clearly, the distance does not provide a direction in which the vehicle has to move. However, it is intuitively clear that one can obtain a direction by using multiple measurements of the distance. Extremum seeking feedback exploits this procedure in a systematic way and can be used for steering dynamical systems to the extremum of an unknown map. Extremum seeking has a long history and has found many applications to diverse problems in control and communications (see Moase, Manzie, Nesic, & Mareels, 2010 and references therein).

In this paper, we provide a novel methodology to analyze extremum seeking systems which differs from commonly used techniques (see e.g. Sanders, Verhulst, & Murdock, 2007). Specifically, this work contains three main contributions.

First, we provide a novel view on extremum seeking by identifying the sinusoidal perturbations in the extremum seeking system as artificial inputs and by writing it in a certain input-affine form. Based on this input-affine form, we derive an approximate system which captures the behavior of the trajectories of the original extremum seeking system. It turns out that the approximate system can be represented by certain Lie brackets of the vector fields in the extremum seeking system. We call this approximate system the Lie bracket system. The proposed methodology is different from results in the existing literature (see e.g. Krstić & Ariyur, 2003, Krstic & Wang, 2000 and Tan, Nešić, & Mareels, 2006).

Second, we establish a theoretic foundation which is based on this novel viewpoint. We prove that the trajectories of a class of input-affine systems with certain inputs are approximated by the trajectories of the Lie bracket systems. Similar results concerning sinusoidal inputs are covered in Kurzweil and Jarnik (1987) and were extended in Gurvits (1992) and Li & Gurvits, 1992 to the class of periodic inputs. In Sussmann and Liu (1991) and Sussmann and Liu (1992) convergence of trajectories of a class of input-affine systems to the trajectories of more general Lie bracket systems was established. These results are closely related to our results. Furthermore, we prove under mild assumptions that semi-global (local) practical uniform asymptotic stability of a class of input-affine systems follows from global (local) uniform asymptotic

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stability of the corresponding Lie bracket systems. These results are based on Moreau and Aeyels (2000, 2003). Summarizing, to the authors best knowledge, the generality of the setup proposed herein was not addressed in the literature before.

Third, we apply the established results to analyze the behavior and the stability properties of extremum seeking vehicles with single-integrator and unicycle dynamics and with static maps. We formulate a multi-agent setup consisting of extremum seeking systems where the individual nonlinear maps of the agents satisfy a certain relationship which assures the existence of a potential function. We use the established theoretical results to show that the set of extrema of the potential function is (locally or semi-globally) practically uniformly asymptotically stable for the multi-agent system. This multi-agent setup is strongly related to game theory and potential games (see Monderer & Shapley, 1996). In the single-agent case, this potential function coincides with the individual nonlinear map. Similar extremum seeking vehicles were analyzed in Zhang, Arnold, Ghods, Siranosian, and Krstić (2007) and Zhang, Siranosian, and Krstić (2007) by using averaging theory (see Khalil, 2002 and Sanders et al., 2007). The authors proposed various extremum seeking feedbacks for different vehicle dynamics and provided a local stability analysis for quadratic maps. Using sinusoidal perturbations with vanishing gains, the authors of Stanković and Stipanović (2009, 2010) were able to extend these results to prove almost sure convergence in the case of noisy measurements of the map. In a slightly different setup the authors of Tan et al. (2006) considered feedbacks which stabilize the extremum of a scalar, dynamic input–output map and established semi-global practical stability of the overall system under some technical assumptions. Multi-agent extremum seeking setups which use similar game-theoretic approaches can be found in Stanković, Johansson, and Stipanović (2012), where the agents seek a Nash equilibrium (see Nash, 1951). The authors proved almost sure convergence of the scheme but without explicit consideration of the global stability properties. A closely related result, which considers the local stability of Nash equilibrium seeking systems, can be found in Frihauf, Krstic, and Basar (2012).

Preliminary results of this work were published in Dürr, Stanković, and Johansson (2011a,b) where the main proofs were omitted. Moreover, the results in this paper are more general.

1.1. Organization

The remainder of this paper is structured as follows. In Section 2 we illustrate the main idea using a simple example. In Section 3 we present theoretical results which link the stability properties of an input-affine system to its Lie bracket system. In Section 4 we apply these results to analyze stability properties of multi-agent extremum seeking systems. Finally, in Section 5 we illustrate the results with examples and give a conclusion in Section 6.

1.2. Notation

$\mathbb{N}_0$ denotes the set of positive integers including zero. $\mathbb{Q}_{++}$ denotes the set of positive rational numbers. The intervals of real numbers are denoted by $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$ and $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, then we write $f(\cdot, y)$ if we consider f as a function of the first argument only and for all $y \in \mathbb{R}^m$. We denote by C^n with $n \in \mathbb{N}_0$ the set of n times continuously differentiable functions and by C^∞ the set of smooth function. The norm $|\cdot|$ denotes the Euclidean norm. The Jacobian of a continuously differentiable function $b \in C^1 : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by

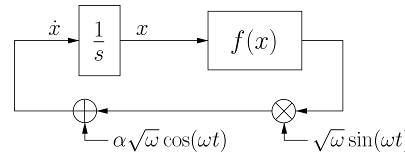

Fig. 1. Basic extremum seeking system.

$$\frac{\partial b(x)}{\partial x} := \begin{bmatrix} \frac{\partial b_1(x)}{\partial x_1} & \cdots & \frac{\partial b_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_m(x)}{\partial x_1} & \cdots & \frac{\partial b_m(x)}{\partial x_n} \end{bmatrix}$$

and the gradient of a continuously differentiable function $J \in C^1 : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla_x J(x) := \left[\frac{\partial J(x)}{\partial x_1}, \ldots, \frac{\partial J(x)}{\partial x_n}\right]^\top$. The Lie bracket of two vector fields $f, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(t, \cdot), g(t, \cdot)$ being continuously differentiable is defined by $[f, g](t, x) := \frac{\partial g(t,x)}{\partial x} f(t, x) - \frac{\partial f(t,x)}{\partial x} g(t, x)$. The a-neighborhood of a set $\mathcal{S} \subseteq \mathbb{R}^n$ with $a \in (0, \infty)$ is denoted by $\mathcal{U}_a^{\mathcal{S}} := \{x \in \mathbb{R}^n : \inf_{y \in \mathcal{S}} |x - y| < a\}$. $\bar{\mathcal{U}}_a^{\mathcal{S}}$ denotes the closure of $\mathcal{U}_a^{\mathcal{S}}$. A function $u : \mathbb{R} \to \mathbb{R}$ is called measurable if it is Lebesgue-measurable. We use $s \in \mathbb{C}$ for the complex variable of the Laplace transformation if not indicated otherwise.

2. Main idea

One simple extremum seeking feedback for static maps is shown in Fig. 1 (see also Krstić & Ariyur, 2003 and Zhang, Siranosian et al., 2007). Suppose that the function $f \in C^2 : \mathbb{R} \to \mathbb{R}$ admits a local, strict maximum at x^* and $\alpha, \omega \in (0, \infty)$.

The extremum seeking system can be written as

$$\dot{x} = \alpha\sqrt{\omega}\cos(\omega t) + f(x)\sqrt{\omega}\sin(\omega t). \tag{1}$$

The main idea is now to identify $\sin(\omega t)$ and $\cos(\omega t)$ as artificial inputs, i.e. $u_1(\omega t) := \cos(\omega t)$ and $u_2(\omega t) := \sin(\omega t)$. Thus, we obtain an input-affine system of the form

$$\dot{x} = b_1(x)\sqrt{\omega}u_1(\omega t) + b_2(x)\sqrt{\omega}u_2(\omega t) \tag{2}$$

with $b_1(x) = \alpha$ and $b_2(x) = f(x)$. Interestingly, if one computes the so called Lie bracket system involving $[b_1, b_2]$, i.e.

$$\dot{z} = \frac{1}{2}[b_1, b_2](z) = \frac{\alpha}{2}\nabla_z f(z), \tag{3}$$

then one sees that this system maximizes f. Having in mind, that trajectories resulting from sinusoidal inputs in (1) can be approximated by trajectories of (3) (see Gurvits, 1992, Kurzweil & Jarnik, 1987, Li & Gurvits, 1992, Sussmann & Liu, 1992) allows us to establish a novel methodology to analyze extremum seeking systems.

The goal of this paper is to generalize this viewpoint to a larger class of extremum seeking systems. We derive a methodology which allows us to analyze a broad class of extremum seeking systems by calculating their respective Lie bracket systems. The procedure can be summarized as follows: Write the extremum seeking system in input-affine form, calculate its corresponding Lie bracket system and prove asymptotic stability of the Lie bracket system which implies practical asymptotic stability for the extremum seeking system.