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Brief paper

On dimensionality reduction and the stability of a class of switched descriptor systems ☆

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ABSTRACT

In this paper we present a dimensionality reduction result for linear descriptor systems. This result is then used to derive stability conditions for special classes of switched descriptor systems. Examples are given to illustrate the efficacy of our stability conditions.

1. Introduction

Descriptor systems belong to a class of dynamical systems that are characterized by both algebraic and differential constraints. Such descriptor systems appear frequently in engineering systems; for example, in the description of interconnected large scale systems, in economic systems (e.g. the fundamental dynamic Leontief model), network analysis (Dai, 1989) and they are also particularly important in the simulation and design of very large scale integrated (VLSI) circuits.

Such systems have been studied widely in both the engineering community, and in the numerical linear algebra community (Campbell, 1982; Kunkel & Mehrmann, 2006), and many properties of such systems are now understood. Until now work on descriptor systems has mainly been interested in characterizing properties such as stability and passivity (Zeheb, Shorten, & Sajja, 2010). Recently, motivated by certain applications, some authors have begun the study of descriptor systems that are characterized by switching between a number of descriptor modes (Liberzon & Trenn, 2009; Trenn, 2009; Zhai & Xu, 2011). The main reason for doing this is that descriptor systems are a rich system class capturing a rich set of dynamic behaviors.

Initial work on switched descriptor systems can be found in Trenn (2009). As was pointed out in this work, switching descriptor systems are particularly challenging as they give rise to behaviors that are found neither in regular switched systems, nor in LTI (linear time invariant) descriptor systems, and generally speaking the behavior of this system class is poorly understood. More specifically, in the study of switched descriptor systems, the evolution of the state is governed not only by the modes of the system but also by the behavior of the state x as the system switches between modes. The latter point is subtle and gives rise to a complex set of behaviors. Despite these difficulties, several authors have already studied the problem of switched descriptor system stability; and conditions for stability have been obtained. In Liberzon and Trenn (2009) the authors focus on dwell time arguments, and on conditions on the "consistency projectors", to obtain stability under arbitrary switching. Similarly, in Zhai and Xu (2011), the authors, under an assumption of a state-dependent switching condition (to avoid impulses), obtain a condition for stability based on commuting vector fields. Our approach in this paper differs from that given in the above papers.

Our motivation is a classical one; namely, we are interested in obtaining conditions that are easily verifiable without resorting

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to complex numerical linear algebra. We succeed in doing that in a special system class; namely, a class of switching systems characterized by rank-1 perturbations, for which a simple continuity assumption on the state at the switching instances is satisfied. An interesting point in our derivation is that these results are obtained using only the simple notion of a full rank decomposition applied to the singular matrix arising in the state space description (Zeheb et al., 2010). Surprisingly, this idea appears to be new in the study of descriptor systems, and in fact gives rise to a powerful dimensionality reduction result that constitutes the main result of our paper.

2. Preliminaries

Consider an LTI descriptor system described by

$$E\dot{x} = Ax, \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$. When E is nonsingular, this system is also described by the normal system $\dot{x} = E^{-1}Ax$. When E is singular, this reflects the fact that both algebraic constraints and differential equations describe the behavior of the system. Since we are interested in systems which are exponentially stable about the origin, we require A to be nonsingular; if A is singular, there are equilibrium states other than zero. The following notions are important in studying descriptor systems.

Stability: The system is said to be stable if all the eigenvalues of the pair (E, A) have negative real parts; an eigenvalue of (E, A) is any complex number λ for which $\det[\lambda E - A] = 0$. When A is nonsingular, (E, A) has no eigenvalues at zero and for $\lambda \neq 0$

$$\det[\lambda E - A] = (-\lambda)^n \det[A] \det[\lambda^{-1}I - A^{-1}E]. \tag{2}$$

From this expression, it is clear that *the eigenvalues of (E, A) are simply the inverse of the non-zero eigenvalues of $A^{-1}E$*. Hence (E, A) *is stable if and only if the non-zero eigenvalues of $A^{-1}E$ have negative real parts*.

Consistency space: When A is invertible, system description (1) is equivalent to

$$x = A^{-1}E\dot{x}. \tag{3}$$

This means that $x(t)$ must always be in the subspace $Im(A^{-1}E)$; hence $\dot{x}(t)$ must be in $Im(A^{-1}E)$ which in turn implies that $x(t)$ must be in $Im((A^{-1}E)^2)$. By induction, we obtain that $x(t)$ is in $Im((A^{-1}E)^k)$ for all $k = 1, 2, \ldots$. Since

$$Im((A^{-1}E)^{k+1}) \subset Im((A^{-1}E)^k)$$

and $\mathbb{R}^n$ has finite dimension n, there exists $k^* \le n$ such that

$$Im((A^{-1}E)^{k^*+1}) = Im((A^{-1}E)^{k^*}); \tag{4}$$

in that case $Im((A^{-1}E)^k) = Im((A^{-1}E)^{k^*})$ for all $k \ge k^*$. Let

$$\mathscr{C} = \mathscr{C}(E, A) := Im((A^{-1}E)^{k^*}). \tag{5}$$

Since $Im((A^{-1}E)^{k^*+1}) = Im((A^{-1}E)^{k^*})$ we see that $A^{-1}E\mathscr{C} = \mathscr{C}$; this means that $A^{-1}E$ is a one-to-one mapping of $\mathscr{C}$ onto itself; hence the kernel of E and $\mathscr{C}$ intersect only at zero (Owens & Debeljkovic, 1985). If we let $\tilde{A}$ be the inverse of the map $A^{-1}E$ restricted to $\mathscr{C}$, then (1), or equivalently (3), is equivalent to

$$\dot{x} = \tilde{A}x. \tag{6}$$

Thus the descriptor system is equivalent to the normal system (6) where $x(t)$ is in $\mathscr{C}$. We call $\mathscr{C}(E, A) = \mathscr{C}$ the consistency space for system (1) or (E, A). Note that $\tilde{A}$ is invertible on $\mathscr{C}$. Also, $\mathscr{C}$ is the set of initial states x_0 for which the system has a solution.

Index: The index of the system is the smallest integer k^* for which (4) holds. If E is singular, we make the following claim where the nullity of E is the dimension of the kernel of E and equals $n - r$ where $r = rank(E)$. *A system is index one if and only if the number of zero eigenvalues of $A^{-1}E$ equals the nullity of E.* To see this, note that the number of zero eigenvalues of $A^{-1}E$ is the algebraic multiplicity of zero as an eigenvalue of $A^{-1}E$ whereas the nullity of E (which equals the nullity of $A^{-1}E$) is the geometric multiplicity of zero as an eigenvalue of $A^{-1}E$. The geometric and algebraic and geometric multiplicities are equal if and only if $A^{-1}E$ and $(A^{-1}E)^2$ have the same nullity; this is equivalent to $Im((A^{-1}E)^2) = Im(A^{-1}E)$, that is, the system is index one.

Switching LTI descriptor systems: The ultimate objective of this work is to analyze the stability of switching descriptor systems described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, \ldots, N\}. \tag{7}$$

We assume throughout this paper that σ is piecewise continuous with a finite number of discontinuities in any bounded time interval. Thus, if σ is continuous at t and $\sigma(t) = i$, the system must satisfy

$$E_i\dot{x}(t) = A_ix(t);$$

hence $x(t)$ must be in the consistency space of (E_i, A_i). To complete the description of a switching descriptor system we must also specify how the system behaves at a point t_* of discontinuity of σ. If σ switches from i to j at t_* then $x(t_*^-) := \lim_{t\to t_*, t<t_*} x(t)$ must be in $\mathscr{C}(E_i, A_i)$ and $x(t_*^+) := \lim_{t\to t_*, t>t_*} x(t)$ must be in $\mathscr{C}(E_j, A_j)$. If $x(t_*^-)$ is not in $\mathscr{C}(E_j, A_j)$ then, one has to have a solution which is discontinuous at t_* and to complete the description one must specify how $x(t_*^+)$ is obtained from $x(t_*^-)$. Commonly, the switching condition on the state can be described by:

$$x(t_*^+) = M_{ji}x(t_*^-) \tag{8}$$

when σ switches from i to j at t_*. See Trenn (2009) for more details on this approach. In the sequel we shall not need explicit calculation of these matrices here; their mere existence will allow us to deduce certain stability properties. Also, switching may be restricted in the sense that one does not switch from i to j at any state $x(t_*^-)$ in $\mathscr{C}(E_i, A_i)$. In this case, the restriction may be described by

$$C_{ji}x(t_*^-) = 0. \tag{9}$$

3. Main result: order reduction for switching descriptor systems

The use of full rank decompositions to reduce a descriptor system was introduced in Zeheb et al. (2010) to deduce certain passivity properties of descriptor systems. Here we extend this idea to obtain a theorem that can be used to study general switching descriptor systems. Before stating this result recall that a pair of matrices (X, Y) is a decomposition of $E \in \mathbb{R}^{n\times n}$ if

$$E = XY^T. \tag{10}$$

If, in addition, X and Y both have full column rank we say that (X, Y) is a full rank decomposition of E. Note that, if (X, Y) is a full rank decomposition of $E \in \mathbb{R}^{n\times n}$ and $rank(E) = r$ then, $X, Y \in \mathbb{R}^{n\times r}$ and $rank(X) = rank(Y) = r$.

Theorem 1 (*Order Reduction*)**.** *Consider a switching descriptor system described by (7) and switching conditions (8)–(9) when σ switches from i to j and suppose that (X_i, Y_i) is a decomposition of E_i with $Y_i \in \mathbb{R}^{n\times r}$ for $i = 1, \ldots, N$. Then, there exist matrices $T_1, \ldots, T_N$ such that the following holds. A function $x(\cdot)$ is a solution to system (7)–(9) if and only if*

$$x(t) = T_{\sigma(t)}z(t) \tag{11}$$