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Max-plus representation for the fundamental solution of the time-varying differential Riccati equation☆

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ABSTRACT

Using the tools of optimal control, semiconvex duality and max-plus algebra, this work derives a unifying representation of the solution for the matrix differential Riccati equation (DRE) with time-varying coefficients. It is based upon a special case of the max-plus fundamental solution, first proposed in Fleming and McEneaney (2000). Such a fundamental solution can extend a particular solution of certain bivariate DREs into the general solution, and the DREs can be analytically solved from any initial condition.

This paper also shows that under a fixed duality kernel, the semiconvex dual of a DRE solution satisfies another dual DRE, whose coefficients satisfy the matrix compatibility conditions involving Hamiltonian and certain symplectic matrices. For the time-invariant DRE, this allows us to make dual DRE linear and thereby solve the primal DRE analytically. This paper also derives various kernel/duality relationships between the primal and time shifted dual DREs, which lead to an array of DRE solution methods. Time-invariant analogue of one of these methods was first proposed in McEneaney (2008).

1. Introduction

The differential Riccati equation (DRE) plays a central role in estimation and optimal control.

An extensive study of algorithms for solving time-invariant and time-varying DREs was carried out by Kenney and Leipnik (1985). These include direct integration, the Chandrasekhar, Leipnik, Davison–Maki, modified Davison–Maki algorithms. Later important developments include a Bernoulli substitution algorithm by Laub (1982), eigenvector decomposition techniques by Oshman and Bar-Itzhack (1985), generalized partitioned solutions and integration free algorithms by Lainiotis (1976), superposition laws developed by Sorine and Winternitz (1985), solutions by Rusnak (1988, 1998). More recently, a fundamental solution based on max-plus algebra and semiconvex duality was proposed by McEneaney (2008).

The purpose of this paper is to present a new representation of the fundamental solution of the time-varying DRE. The fundamental solution allows us to efficiently compute a general solution starting from different initial conditions. This representation uses the max-plus techniques and is inspired from McEneaney (2008), but it extends the solution to the time-varying DRE and simplifies the treatment by not using the semiconvex duality for the main result. In process, it derives the special case of the max-plus fundamental solution first proposed by Fleming and McEneaney in Fleming and McEneaney (2000), for the linear-quadratic problem. It also shows that such a fundamental solution is bivariate quadratic and describes the algorithm to compute the same. It shows that evolution of a DRE under the max-plus fundamental solution is also a semiconvex dual transformation with a suitable kernel. Further it shows that the semiconvex dual transformation of a DRE, satisfies another DRE. It then derives the matching conditions between the coefficients and duality kernel relationships between primal and dual solutions at different times.

The DRE solution itself is similar in structure to the previous algorithms. Specifically, the fundamental solution computation requires integration of three ODEs similar to the forward formulae in Lainiotis (1976) and 1-representation addition formula in Sorine and Winternitz (1985). Still, the max-plus framework presented here is unifying and general. e.g. partitioned formulae for the forward and backward time-varying DREs in Lainiotis (1976), time-invariant DRE solutions in Leipnik (1985), McEneaney (2008), and Rusnak (1988) can be derived as special cases of a single framework. In addition, it is known that such algorithms work well for the stiff time-varying DREs and long time horizons without any computational difficulties, unlike the time-marching algorithms or the Davison–Maki algorithm.

2. Optimal control problem

We consider the matrix *differential Riccati equation* (DRE) of the form

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$$-\dot{p}(t) = A(t)'p(t) + p(t)A(t) + C(t) + p(t)\Sigma(t)p(t) \quad (1)$$

given the boundary condition $p(T)$ at time T. Here $t \in (-\infty, T]$, $A(t) \in \mathbb{R}^{n\times n}$, $p(t)$, $C(t)$, $\Sigma(t)$ are square and symmetric $n \times n$ matrices and $\Sigma(t) = \sigma(t)\sigma(t)' \succeq 0$ where $\sigma(t)$ is $n \times m$ matrix. Note that the notation A' denotes the transpose of matrix A. It is well known that above DRE arises in the optimal control problem with linear dynamics

$$\dot{\xi}_s = f_s(\xi_s, u_s) \doteq A(s)\xi_s + \sigma(s)u_s, \quad \xi_t = x \in \mathbb{R}^n \quad (2)$$

and the following payoff function consisting of the integral and terminal payoffs,

$$J_t^T(x, u) \doteq \int_t^T l_s(\xi_s, u_s)\,ds + \phi(\xi_T), \quad \text{where} \quad (3)$$

$$l_s(\xi, \upsilon) \doteq \frac{1}{2}\xi'C(s)\xi - \frac{1}{2}|\upsilon|^2 \quad \text{and} \quad (4)$$

$$\phi(\xi) \doteq \frac{1}{2}\xi'p(T)\xi, \quad \text{for all } \xi \in \mathbb{R}^n, \upsilon \in \mathbb{R}^m. \quad (5)$$

Then, the optimal payoff or the value function is also quadratic given by,

$$V(t, x) \doteq \sup_{u\in L_2(t,T)} J_t^T(x, u) = \frac{1}{2}x'p(t)x, \quad (6)$$

and $p(t)$ follows the DRE (1). In order to ensure the existence and the regularity of the value function and for the development to follow, we make following assumptions.

> Let $\bar{T} < T$. We assume that $\forall t \in (\bar{T}, T]$, $A(t)$, $C(t)$, $\Sigma(t) \in \mathbb{R}^{n\times n}$ are piecewise continuous, locally bounded functions of time t. Moreover, $\Sigma(t)$, $C(t)$ are symmetric and $\Sigma(t) \doteq \sigma(t)\sigma(t)' \succeq 0$. We also assume that the underlying dynamic system (2) is controllable. Since the DRE may exhibit finite time blowup, we assume that for $t \in (\bar{T}, T]$ there exists a solution of DRE (1) with the terminal condition $p(T) = P_T$. We denote this solution by P_t for the ease of notation. (7)

Now we shall obtain the fundamental solution for DRE (1) through the following generalization of the above optimal control problem. We assume the same dynamics as (2), and assume the following payoff function in which the integral payoff l_s is as defined in (4) and the terminal payoff is parametrized as below by $z \in \mathbb{R}^n$,

$$\begin{aligned} J_t^T(x, u; z) &\doteq \int_t^T l_s(\xi_s, u_s)\,ds + \phi(\xi_T; z), \quad \text{where} \\ \phi(\xi; z) &\doteq \phi^z(\xi) \doteq \frac{1}{2}\xi'P_T\xi + \xi'S_Tz + \frac{1}{2}z'Q_Tz \end{aligned} \quad (8)$$

for all $\xi \in \mathbb{R}^n$.

Note that for the terminal payoff, we have reused the notation and (5) is the special case of $\phi(\xi; 0)$ above, when $p(T) = P_T$.

The optimal payoff or the value function is defined as

$$V(t, x; z) \doteq V_t^z(x) \doteq \sup_{u\in L_2(t,T)} J_t^T(x, u; z) \quad (9)$$

for all $x, z \in \mathbb{R}^n$ and $t \in (\bar{T}, T]$. Now we state an important theorem regarding such a value function, which is proved in the Appendix.

Theorem 1. *Assume (7), and assume that P_T, Q_T are symmetric matrices and S_T is invertible. Then for any $z \in \mathbb{R}^n$, the value function (9) is given by.*

$$V(t, x; z) = \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz \quad (10)$$

where P_t, S_t, Q_t evolve as per

$$\begin{aligned} -\dot{P}_t &= A(t)'P_t + P_tA(t) + C(t) + P_t\Sigma(t)P_t \\ -\dot{S}_t &= (A(t) + \Sigma(t)P_t)'S_t \\ -\dot{Q}_t &= S_t'\Sigma(t)S_t, \end{aligned} \quad (11)$$

and satisfy the boundary conditions P_T, S_T and Q_T, respectively, at time $t = T$. Further, the optimal control at a state $\tilde{\xi}_s$ at time s is

$$\tilde{u}_s = \sigma(s)'(P_s\tilde{\xi}_s + S_sz), \quad (12)$$

and the corresponding optimal trajectory $\tilde{\xi}$, starting at $\tilde{\xi}_t = x$ and evolving as per the control (12) satisfies

$$S_{t_2}'\tilde{\xi}_{t_2} + Q_{t_2}z = S_{t_1}'\tilde{\xi}_{t_1} + Q_{t_1}z, \quad (13)$$

for $\bar{T} < t_1 < t_2 \le T$. Further, $Q_{t_1} - Q_{t_2} \succ 0$ and S_t is invertible for $t \in (\bar{T}, T]$.

Proof. Lemmas 20–23 in the Appendix, together prove the above result. □

Remark 2. Since S_{t_1} and S_{t_2} are invertible, (13) suggests a one–one and onto relation between start and end of optimal trajectories, ξ_{t_1} and ξ_{t_2} for all z. Thus $\forall y \in \mathbb{R}^n$ there exists a $x = S_{t_2}^{-1'}(S_{t_1}'y + (Q_{t_1} - Q_{t_2})z)$ such that optimal trajectory $\tilde{x}_t$ starting at $\tilde{x}_{t_1} = x$, ends with y. Thus every $y \in \mathbb{R}^n$ is an optimal point for some initial condition.

3. Max-plus fundamental solution

Given $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, system trajectory starting at $\xi_{t_1} = x$ and a general terminal payoff function $\psi : \mathbb{R}^n \to \mathbb{R}$, let us define the operator,

$$\mathcal{S}_{t_1}^{t_2}[\psi](x) \doteq \sup_{u\in L_2(t_1,t_2)} \int_{t_1}^{t_2} l_s(\xi_s, u_s)\,ds + \psi(\xi_{t_2}). \quad (14)$$

We can restate (9) and (8) using above operator. Noting that $V_T^z(x) = \phi^z(x)$, as defined in (8), we have for all $t \in (\bar{T}, T]$

$$V_t^z(x) = \mathcal{S}_t^T[\phi^z](x) = \mathcal{S}_t^T[V_T^z](x).$$

It is well known that operators $\mathcal{S}_{t_1}^{t_2}$ form a semigroup. Thus if $t_1 \le t \le t_2 \le T$, then $\mathcal{S}_{t_1}^{t_2}[\psi] = \mathcal{S}_{t_1}^{t}[\mathcal{S}_t^{t_2}[\psi]]$, which is the celebrated dynamic programming principle for this problem. That is with $t_2 = T$,

$$\begin{aligned} V_{t_1}^z(x) &= \mathcal{S}_{t_1}^T[\phi^z](x) = \mathcal{S}_{t_1}^t[\mathcal{S}_t^T[\phi^z]](x) = \mathcal{S}_{t_1}^t[V_t^z](x) \\ &= \sup_{u\in L_2(t_1,t)} \int_{t_1}^{t} l_s(x_s, u_s)\,ds + V_t^z(\xi_t). \end{aligned} \quad (15)$$

If we define $a \oplus b \doteq \max(a, b)$ and $a \otimes b \doteq a + b$, then it is well known that $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ forms a commutative semifield which is referred to as the max-plus algebra (see Baccelli, Cohen, Olsder, & Quadrat, 1992; Helton & James, 1999; Litvinov & Maslov, 1998, for a fuller discussion). We can extend this algebra to functions so as to define the max-plus vector space. Let $[a\oplus b](x) = \max(a(x), b(x))$ and $a(x) \otimes k = a(x) + k$, where $a, b : \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{R}$. Maslov (1987) proved that the above semigroup is linear in max-plus algebra. Thus using above notation

$$\begin{aligned} \mathcal{S}_{t_1}^T[\psi_1 \oplus \psi_2](x) &\doteq \mathcal{S}_{t_1}^{t_2}[\max(\psi_1, \psi_2)](x) \\ &= \max\{\mathcal{S}_{t_1}^{t_2}[\psi_1](x), \mathcal{S}_{t_1}^{t_2}[\psi_2](x)\} \\ &\doteq \mathcal{S}_{t_1}^T[\psi_1](x) \oplus \mathcal{S}_{t_1}^T[\psi_2](x) \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{t_1}^T[k \otimes \psi_1](x) &\doteq \mathcal{S}_{t_1}^{t_2}[k + \psi_1](x) \\ &= k + \mathcal{S}_{t_1}^{t_2}[\psi_1](x) \doteq k \otimes \mathcal{S}_{t_1}^T[\psi_1](x). \end{aligned}$$