Automatica 47 (2011) 1677–1684

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Root mean square gain of discrete-time switched linear systems under dwell time constraints[☆]

P. Colaneri[a,*], P. Bolzern[a], J.C. Geromel[b]

[a] Politecnico di Milano, Dipartimento di Elettronica e Informazione, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

[b] DSCE/School of Electrical and Computer Engineering, UNICAMP, 13083-852, Campinas, SP, Brazil

ARTICLE INFO

Article history:
Received 27 October 2009
Received in revised form
20 September 2010
Accepted 25 January 2011
Available online 21 March 2011

Keywords:
Switched linear systems
$\mathcal{H}_\infty$ control
Root mean square gain
Dwell time
LMI

ABSTRACT

This paper deals with discrete-time switched linear systems and considers the problem of computing an upper bound to the dwell time ensuring a pre-specified root mean square (RMS) gain. As a natural consequence of treating general systems of this class in terms of the order and the number of subsystems, only sufficient conditions are worked out. They depend on the complete separation of the stabilizing and anti-stabilizing solutions of the algebraic Riccati equation associated to each subsystem. Moreover, as positive features, it is shown that the dwell time preserving the specification can be calculated through linear matrix inequalities (LMIs) and line search, being thus numerically solvable in polynomial time, and this allows the treatment of stable switched linear systems which do not admit a common Lyapunov function. The case of a guaranteed RMS gain for arbitrary switching signals is also addressed. A simple academic example constituted by three subsystems of third order is included for illustration.

1. Introduction

Hybrid and switched dynamic systems have received a great deal of attention in recent decades. The stability analysis of continuous-time switched linear systems has been addressed by many authors; see, for example, Branicky (1998), Hockerman-Frommer, Kulkarni, and Ramadge (1998), Hespanha and Morse (1999), Hespanha (2004), Johansson and Rantzer (1998), Wirth (2005) and Ye, Michel, and Hou (1998). General results on this topic are presented in the book Blanchini and Miani (2008) and in the survey papers Margaliot (2006) and Shorten, Wirth, Mason, Wulff, and King (2007). More specifically, in Hespanha (2004), the interested reader can find a collection of results on uniform stability of switched systems. The reader is also requested to see DeCarlo, Branicky, Pettersson, and Lennartson (2000), Liberzon (2003) and Liberzon and Morse (1999) for a fairly complete review of the stability of continuous-time switched linear systems, where special attention is given to the case of switching between two linear systems. For control synthesis, see Geromel, Colaneri, and Bolzern (2008), Ishii, Basar, and Tempo (2005), Morse (1996) and Wicks and DeCarlo (1997). In this paper, the stability conditions for discrete-time linear switched systems provided in Geromel and Colaneri (2006) are used. They help in determining an upper bound to the minimum dwell time that preserves stability and are expressed in terms of linear matrix inequalities (LMIs) plus a scalar variable, being thus solvable in polynomial time. For discrete-time switched systems, see also Lin and Antsaklis (2008), Ji, Wang, and Xie (2005), Zhai, Hu, Yasuda, and Michel (2002), Daafouz and Bernussou (2001), Liberzon (2009), Xie and Wang (2003) and Zhai (2001), where several results on stability analysis and control synthesis are presented.

This paper deals with the discrete-time switched linear system with the following state space representation, evolving from zero initial condition,

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t), \qquad x(0) = 0 \tag{1}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t), \tag{2}$$

defined for all integers $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the exogenous input, $y(t) \in \mathbb{R}^r$ is the output, and $\sigma(t) : t \geq 0 \to \{1, \ldots, N\}$ is the switching rule. In addition to the papers already cited, in Lee and Dullerud (2006a,b), nonconservative stability analysis, RMS gain calculation, and synthesis of switched controllers are provided by using path-dependent Lyapunov functions. The focus of the present paper is on the determination of the RMS gain by considering some

☆ This research was supported by a grant from "Conselho Nacional de Desenvolvimento Científico e Tecnológico-CNPq", Brazil. The material in this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Yasumasa Fujisaki under the direction of Editor Roberto Tempo.

* Corresponding author.
E-mail addresses: colaneri@elet.polimi.it (P. Colaneri), bolzern@elet.polimi.it (P. Bolzern), geromel@dsce.fee.unicamp.br (J.C. Geromel).

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.02.035

classes of (nonperiodic) switching rules characterized by a given minimum dwell time d. Arbitrary switching is also considered by enforcing $d = 1$, in which case the sufficient conditions of Daafouz and Bernussou (2001) are obtained. Compared with the procedure in Lee and Dullerud (2006b), applicable to more general classes of switching signals, the conditions provided in the present paper, though generally more conservative, are easier to check, especially for large values of N and d. Notably, the numerical complexity of the proposed algorithm does not depend on d. This is a consequence of self-contained results on the difference Riccati equation, in particular the extension to discrete-time systems of the ordering property valid for continuous-time ones provided in Hespanha (2003). For the continuous-time counterpart of RMS gain calculation, see also Geromel and Colaneri (2010).

Let $\mathcal{D}_d$ be the set of all switching policies with dwell time greater than or equal to d steps, that is, the set of all $\sigma(t)$ for which the time interval between successive switchings satisfies $t_{k+1} - t_k \geq d > 0$. Given a positive value of the attenuation level γ, the main goal is to provide conditions for a guaranteed RMS gain with dwell time specification, i.e.

$$J(\sigma) := \sup_{w \in l_2} \|y\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0, \tag{3}$$

for all $\sigma \in \mathcal{D}_d$. The complete solution to (3) is difficult to obtain, since the global solution to the optimal control problem

$$\sup_{\sigma \in \mathcal{D}_d} J(\sigma) = \sup_{\sigma \in \mathcal{D}_d, w \in l_2} \|y\|_2^2 - \gamma^2 \|w\|_2^2$$

is difficult to calculate due to the algebraic structure of the set $\mathcal{D}_d$ for any dwell time $d \in \mathbb{N}$ fixed. Blondel and Tsitsiklis (2000) and Xu and Antsaklis (2004) give an idea of the difficulties to be faced if we want to solve optimal control problems of this class with the state space dimension n and the number of subsystems N taken arbitrarily; see also Margaliot and Hespanha (2008). Hence, we will concentrate on the determination of a suboptimal solution and validate its quality through an example. The conservativeness of the proposed conditions to assure the validity of (3) may be tested by imposing that (3) has to hold for a subset $\mathcal{P}_d \subset \mathcal{D}_d$ composed of all periodic switching policies with period $d > 0$. Moreover, the related conditions appearing in Colaneri and Geromel (2008), based on worst-case input determination, are adequately addressed and corrected.

Hence, our main interest is on the numerical determination of the function

$$d_*(\gamma) = \inf_{d>0} \{ d : J(\sigma) \leq 0 \; \forall \, \sigma \in \mathcal{D}_d \} \tag{4}$$

that gives the minimum dwell time associated to each prescribed RMS gain. Since, as already mentioned, this function is not simple to calculate exactly, our main purpose is to determine its bounds $d_p(\gamma) \leq d_*(\gamma) \leq d(\gamma)$, where the upper bound follows from sufficient conditions for $J(\sigma) \leq 0 \; \forall \sigma \in \mathcal{D}_d$ to hold, and the lower bound is obtained from (4) with $\mathcal{D}_d$ replaced by the set of all piecewise periodic functions $\mathcal{P}_d$.

The notation is standard. Capital letters denote matrices, small letters denote vectors, and small Greek letters denote scalars. For matrices or vectors, $(')$ indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (nonnegative definite). The sets of real and natural numbers including zero are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. The set $\mathbb{K}$ is defined as $\mathbb{K} = \{1, \ldots, N\}$. The squared norm of a trajectory $z(t)$ defined for all $t \geq 0$ equals $\|z\|_2^2 = \sum_{t=0}^{\infty} z(t)'z(t)$. All trajectories with finite norm, that is $\|z\|_2^2 < \infty$, characterize the set l_2. For the sake of simplifying the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. The convex combination of matrices with the same dimension $\{F_1, \ldots, F_N\}$ is denoted by $F_\lambda = \sum_{j=1}^{N} \lambda_j F_j$, where λ belongs to the unitary simplex Λ composed by all nonnegative vectors $\lambda \in \mathbb{R}^N$ such that $\sum_{j=1}^{N} \lambda_j = 1$.

2. Preliminaries

This section is entirely devoted to the analysis of the $\mathcal{H}_\infty$ problem for linear time-invariant systems and to provide some properties of the difference and algebraic Riccati equations to be extensively used in what follows. The discrete-time $\mathcal{H}_\infty$ problem can be stated as

$$\sup_{w \in l_2} \sum_{t=0}^{\infty} \big(y(t)'y(t) - \gamma^2 w(t)'w(t) \big), \tag{5}$$

subject to

$$x(t+1) = Ax(t) + Bw(t), \qquad x(0) = x_0$$
$$y(t) = Cx(t) + Dw(t).$$

All indicated matrices of compatible dimensions and the initial state $x_0 \in \mathbb{R}^n$ are given. Under mild standard assumptions on the open-loop system, namely stability, minimality of the state space representation, reachability, and observability, this problem can be solved with no difficulty, and its optimal solution is provided by the positive definite stabilizing solution of the algebraic Riccati equation. As will be clear in what follows, we have an interest in rewriting (5) in the equivalent form

$$\sup_{w \in l_2} \left\{ \sum_{t=t_k}^{t_{k+1}-1} \big(y(t)'y(t) - \gamma^2 w(t)'w(t) \big) + v_{k+1}(x(t_{k+1})) \right\} = v_k(x(t_k)), \tag{6}$$

where $t_{k+1} \geq t_k + 1$ are successive time instants for all $k \in \mathbb{N}$ starting from $t_0 = 0$. The importance of this dynamic programming recursive equation is that $v_0(x_0)$ provides the optimal cost of (5). In addition, if in Eq. (6) the sign $=$ is replaced by the sign $\leq$, then $v_0(x_0)$ becomes an upper bound of the optimal cost of (5). Taking into account that the dynamic system is time-invariant, assuming that the difference Riccati equation

$$\Pi(t) = A'\Pi(t+1)A + C'C + \big(A'\Pi(t+1)B + C'D\big) \times \big(\gamma^2 I - D'D - B'\Pi(t+1)B\big)^{-1} \big(A'\Pi(t+1)B + C'D\big)' \tag{7}$$

admits a positive definite solution in the time interval $[0, \tau_k]$, where $\tau_k = t_{k+1} - t_k > 0$, then $Z_k = \Pi(0)$ determined from the final condition $Z_{k+1} = \Pi(\tau_k)$ allows us to conclude that the quadratic function $v_k(x) = x'Z_k x$ solves the recursive equation (6) and by consequence (5). Indeed, in this case, it is known that $Z_k = P$ for all $k \in \mathbb{N}$, where $P \in \mathbb{R}^{n \times n}$ is the positive definite stabilizing solution of the associated algebraic Riccati equation. Hence, in the context of switched linear systems, the existence of a solution of the $\mathcal{H}_\infty$ difference Riccati equation is a central and important issue. The associated algebraic Riccati equation is

$$\Pi = A'\Pi A + C'C + \big(A'\Pi B + C'D\big) \times \big(\gamma^2 I - D'D - B'\Pi B\big)^{-1} \big(A'\Pi B + C'D\big)'. \tag{8}$$

We say that a solution of (7) or (8) is feasible whenever it satisfies $\gamma^2 I - D'D - B'\Pi(t+1)B > 0$ for all $t \in [0, \tau)$ or $\gamma^2 I - D'D - B'\Pi B > 0$, respectively. For simplicity, let us introduce the notation

$$\Delta(t+1) = \big(\gamma^2 I - D'D - B'\Pi(t+1)B\big)^{-1}$$
$$L(t+1) = \Delta(t+1)\big(A'\Pi(t+1)B + C'D\big)'$$
$$H(t+1) = A + BL(t+1).$$

Similarly, for the algebraic Riccati equation (8), let $\Delta = \big(\gamma^2 I - D'D - B'\Pi B\big)^{-1}$, $L = \Delta \big(A'\Pi B + C'D\big)'$, and $H = A + BL$.

We are particularly interested in two positive definite feasible solutions of the algebraic Riccati equation (8), denoted P and P_a, referred to as the stabilizing and anti-stabilizing solutions, respectively. With a little abuse of notation, we still use (Δ, L, H)