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Brief paper

Rational approximations in the simulation and implementation of fractional-order dynamics: A descriptor system approach☆

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ABSTRACT

This paper deals with issues related to the use of rational approximations in the simulation of fractional-order systems and practical implementations of fractional-order dynamics and controllers. Based on the mathematical formulation of the problem, a descriptor model is found to describe the rational approximating model. This model is analyzed and compared with the original fractional-order system under the aspects which are important in their simulation and implementation. From the results achieved, one can determine in what applications the use of rational approximations would be unproblematic and in what applications it would lead to fallacious results. In order to clarify this point, some examples are presented in which the effects of using rational approximations are investigated.

1. Introduction

The analog realization of fractional-order systems plays an important role in practical applications of these systems in different fields such as the synthesis of fractional-order controllers (Monje, Vinagre, Feliu, & Chen, 2008; Podlubny, 1999a; Tavazoei, Haeri, Jafari, Bolouki, & Siami, 2008) or the implementation of fractional-order oscillators (Oustaloup, 1981; Radwan, El-Wakil, & Soliman, 2008; Radwan, Soliman, & El-Wakil, 2008). One way for analogue realization of fractional-order systems is to use special electrical elements which are known as fractances. In fact, a fractance is an electrical element with non-integer-order impedance (Le Mehaute & Crepy, 1983). In some pioneering works, the fractance is constructed by designing an electrical circuit which consists of infinite resistors and capacitors (Nakagawa & Sorimachi, 1992; Oldham & Zoski, 1983). In practice, the implementation of an electrical circuit with an infinite number of elements is impossible. Therefore, the designed circuits are truncated. Truncation of the circuit in practical applications implies that an integer-order filter is used instead of the real fractance. Another approach to design a fractance is to use fractional capacitors. To fabricate a fractional capacitor, the use of some electrolyte processes or of other material with fractal structures has been proposed in the literature (Biswas, Sen, & Dutta, 2006; Jesus & Machado, 2009), but it seems that the proposed technologies take time to be developed and widely used in practical applications. This means that the use of integer-order filters is still the primary way of analogue realization of a fractional-order element (Petras, Podlubny, O'Leary, Dorcak, & Vinagre, 2002). On the other hand, rational approximations of fractional operators have been used in many papers to simulate a fractional-order system. Since any approximation naturally has some limitations to describe its original counterpart, its usage should be performed with some care and consideration. This paper deals with some clarifications about using such approximations in simulations or practical implementations of a fractional-order system. Based on the outcome of the paper, one can recognize in what applications the use of rational approximations would be allowable and in what applications it would not be admissible. This paper is a sequel to the previously published results in Tavazoei and Haeri (2007) and Tavazoei, Haeri, Bolouki, and Siami (2008).

The paper is organized as follows. Section 2 briefly describes the background of the problem. In Section 3, first the problem is formulated and then a descriptor representation is found for the approximated model. The rest of Section 3 is devoted to analyze the model found. The results of Section 3 are further clarified by presenting some examples in Section 4. Finally, the paper is concluded in Section 5.

2. Preliminaries

By extending the concepts of the ordinary integral and derivative, fractional integral and derivative operators have been defined in the literature. Different versions of these definitions

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can be found in Oldham and Spanier (1974), Samko, Kilbas, and Marichev (1993), and Podlubny (1999b). Also, the geometric, physical and probabilistic interpretations of fractional-order integration and differentiation have been presented in Podlubny (2002) and Machado (2003). Fractional-order systems as a generalization of traditional integer-order systems can be described by differential equations containing fractional derivatives. For example, a fractional-order linear time-invariant system can be defined in the following state space like form:

$$\begin{pmatrix} \mathrm{d}^{\alpha_1}x_1/\mathrm{d}t^{\alpha_1} \\ \mathrm{d}^{\alpha_2}x_2/\mathrm{d}t^{\alpha_2} \\ \vdots \\ \mathrm{d}^{\alpha_n}x_n/\mathrm{d}t^{\alpha_n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{1}$$

where $0 < \alpha_i \leq 1$ and $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ refers to smooth fractional derivation (Matignon, 1996). As a more general form of (1), a fractional-order non-linear time-invariant system may be defined as follows:

$$\mathrm{d}^{\alpha_i}x_i/\mathrm{d}t^{\alpha_i} = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{2}$$

Two prevalent approaches to determine the responses of fractional-order systems (1) and (2) are to solve the fractional-order differential equations involved numerically and to approximate the fractional-order system using fractional operator approximations. The results obtained from the first approach are more reliable than the results given by the second approach; however, due to the long memory characteristics of fractional-order systems, the use of the methods in the first category requires fairly long simulation times (Tavazoei & Haeri, 2007). This drawback is probably the main reason why the second approach has been used in the literature (Aoun, Malti, Levron, & Oustaloup, 2004; Hartley, Lorenzo, & Qammer, 1995). To find a rational approximation of a fractional-order operator, many methods such as Matsuda's method (Matsuda & Fujii, 1993), Oustaloup's method (Oustaloup, Levron, Mathieu, & Nanot, 2000), and Charef's method (Charef, 2006) have been proposed. To simulate a fractional-order system defined by (1) or (2) using rational approximations, first the fractional-order equations of the system are transformed to the frequency domain representation and then the Laplace transforms of the fractional integral operators are replaced by their rational approximations. The approximated equations in the frequency domain are then transformed back into the time domain. The resulting ordinary differential equations can be solved numerically by applying well-known numerical methods such as Euler or Runge–Kutta methods.

On the other hand, the fractional-order system (1) can be implemented by a network of resistance and fractional capacitors. As mentioned in the previous section, in practice the non-integer-order impedances of fractional capacitors are usually replaced by some filters that approximate the desired impedance. By using these filters, the fractional-order system (1) is approximately implemented.

The problem of simulating or implementing a fractional-order system using rational approximations is formulated in the next section. In addition, the approximated model is determined and analyzed.

3. Analysis of the approximated model

Suppose that the rational approximations of the fractional operators $1/s^{\alpha_i}$ for $i = 1, 2, \ldots, n$ are described by the following proper transfer functions:

$$\frac{1}{s^{\alpha_i}} \approx G_i(s) = \frac{Q_i(s)}{P_i(s)} = \frac{q_{m,i}s^m + q_{m-1,i}s^{m-1} + \cdots + q_{1,i}s + q_{0,i}}{p_{m,i}s^m + p_{m-1,i}s^{m-1} + \cdots + p_{1,i}s + p_{0,i}}, \tag{3}$$

where at least one of the $p_{m,i}$ coefficients for $i = 1, 2, \ldots, n$ is non-zero. In other words, m is the maximum order of the approximating filters. Without loss of generality, in this paper it is assumed that $\deg(P_n(s)) \leq \deg(P_{n-1}(s)) \leq \cdots \leq \deg(P_1(s)) = m$. The approximated model is studied in the two subsequent sections for fractional-order linear and non-linear time-invariant systems, respectively.

3.1. Approximated model for linear system (1)

For the given system (1) and with the approximating filters defined in (3), the approximated model can be described by the following differential equations:

$$\sum_{r=0}^{m} p_{r,i}x_i^{(r)} = \sum_{r=0}^{m} \left(q_{r,i}\left(\sum_{k=1}^{n} a_{ik}x_k\right)\right)^{(r)}, \quad i = 1, 2, \ldots, n. \tag{4}$$

This model can also be described by a higher-order descriptor realization given as

$$A_m x^{(m)} + A_{m-1}x^{(m-1)} + \cdots + A_1\dot{x} + A_0 x = 0, \tag{5}$$

where

$$A_i = \mathrm{diag}(p_{i,1}, p_{i,2}, \ldots, p_{i,n}) - \mathrm{diag}(q_{i,1}, q_{i,2}, \ldots, q_{i,n})A.$$

The original system (1) has only one fixed point placed at the origin provided that the matrix $A = [a_{ij}]_{n\times n}$ is non-singular. The approximated model also has a single fixed point at the origin when the matrix A_0 is non-singular.

The higher-order descriptor model (5) can be converted to the following first-order descriptor model: (Duan, 2006)

$$E\dot{z} = Mz, \tag{6}$$

where $z = \begin{pmatrix} x^{\mathrm{T}} & \dot{x}^{\mathrm{T}} & \cdots & \left(x^{(m-1)}\right)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$, and

$$E = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & \vdots \\ \vdots & \vdots & I_n & 0 \\ 0 & \cdots & 0 & A_m \end{pmatrix} \quad \&$$

$$M = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ -A_0 & -A_1 & \cdots & -A_{m-1} \end{pmatrix}.$$

The descriptor model (6) is said to be regular if $\Delta(s) = \det(sE - M)$ is not identically zero. The regularity of model (6) guarantees the existence and uniqueness of the solution $z(t)$ for a given initial condition $Ez(0)$ (Campbell, 1980). It is well known that the response of a descriptor linear system such as (6) may contain impulsive terms. The solution will be impulsive-free if $\deg(\det(sE - M)) = \mathrm{rank}(E)$ (Dai, 1989). Therefore, the approximated model will be impulsive-free if

$$\deg(\det(sE - M)) = n(m-1) + \mathrm{rank}(A_m). \tag{7}$$

The zeros of $\Delta(s) = \det(sE - M)$ are finite poles of system (6). It is clear that, if matrix E is non-singular, the poles of system (6) will be eigenvalues of matrix $E^{-1}M$. System (6) is asymptotically stable if all finite poles, i.e. finite eigenvalues of the pencil $sE - M$, have real parts less than zero (Dai, 1989). The finite poles of system (6) specify the exponential modes of this system. Hence, the solution of the approximated model can at most have $n(m-1) + \mathrm{rank}(A_m)$ exponential modes. Based on the Weierstrass–Kronecker theorem (Dai, 1989), there exist $mn \times mn$ non-singular matrices P and Q such that

$$P(sE - M)Q = \begin{pmatrix} sI_d - J & 0 \\ 0 & sN - I_{mn-d} \end{pmatrix}, \tag{8}$$