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Brief paper

Factorization of multipliers in passivity and IQC analysis ☆

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ABSTRACT

Multipliers are often used to find conditions for the absolute stability of Lur'e systems. They can be used either in conjunction with passivity theory or within the more recent framework of integral quadratic constraints (IQCs). We compare the use of multipliers in both approaches. Passivity theory requires that the multipliers have a canonical factorization and it has been suggested in the literature that this represents an advantage of the IQC theory. We consider sufficient conditions on the nonlinearity class for the associated multipliers to have a canonical factorization.

1. Introduction

The use of open-loop properties, such as applying the small gain theorem as well as the passivity theorem, in order to find absolute stability conditions for the Lur'e problem (see Fig. 1) is a common tool in nonlinear systems theory. In this problem the stability of a linear time-invariant (LTI) system, G, in a feedback interconnection with a nonlinear system, ϕ, is studied. Decoupling the linear and nonlinear parts reduces the complexity of the problem and allows a solution in terms of simple conditions on the linear part. An essential feature of this method is that stability is guaranteed for any nonlinearity ϕ within an entire class of nonlinearities Φ.

Historically, a first general solution for a specific class of nonlinearities was given by Popov (1961); his result is generalized in Yakubovich (1967) for multivariable systems (see Heath and Li (2009) and references therein for different multivariable cases). The circle criterion was developed by several authors simultaneously, but a pair of papers can be highlighted (Zames, 1966a,b). In Zames (1966a), the definition of input–output stability using extended spaces, as proposed by Sandberg (1964), is used and the small gain and passivity theorems are established. In Zames (1966b) the circle and Popov criteria are obtained as applications of these theorems. In the proof of the Popov criterion in Zames (1966b), the abstract concept of multiplier is interpreted as a loop transformation, see Fig. 2.

The multiplier is an artificial system that is introduced into the loop together with its inverse. Roughly speaking, an excess of positivity in the nonlinear part is exploited to redress a deficiency of positivity in the linear part. Passivity theory requires systems to be causal, but restricting the analysis to linear causal multipliers, i.e. systems without poles in the right half plane, leads to severe constraints on the choice of the phase. In Zames and Falb (1968) a factorization condition on non-causal multipliers is proposed to overcome this restriction and recover causality in the loop elements (see Fig. 3 and Remark 2.7).

The factorization condition on the multiplier is given by

$$M = M_- M_+ \quad (1)$$

where M_- and M_+ are invertible and M_+, M_+^{-1}, M_-^*, and M_-^{*-1} are causal and have finite gain. For the Lur'e problem where one part of the loop is LTI it is natural to restrict the multipliers themselves to be LTI. For a linear operator this is referred to as the canonical factorization (see Section 2.2). Some special cases of this factorization, e.g. spectral factorization, inner–outer factorization and J-spectral factorization, have been used in $\mathbf{H}_\infty$ control theory (Francis, 1987). The conditions for the existence of this factorization are summarized in the monograph (Bart, Gohberg, Kaashoek, & Ran, 2010) which takes an operator theoretical approach. In Goh (1996), an equivalent result was found from a control systems perspective. Only a few papers, for instance, Chou, Tits, and Balakrishnan (1999), have used these results for control systems analysis.

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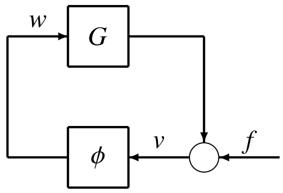

Fig. 1. Lur'e problem.

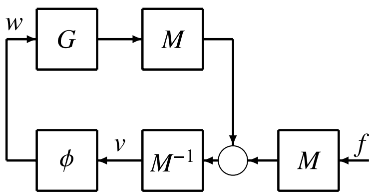

Fig. 2. Multiplier transformation: stability of this systems implies stability of the original system in Fig. 1.

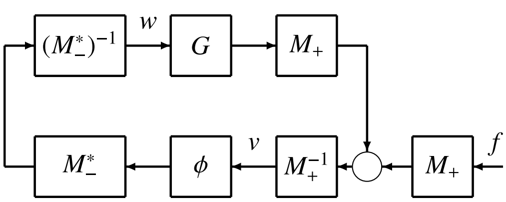

Fig. 3. The factorization (1) of a non-causal multiplier ensures that $M_+G(M_-^*)^{-1}$ and $M_-^*\phi M_+^{-1}$ are causal.

In the multiplier approach the properties of a class Φ of positive nonlinearities ϕ are used to find the corresponding class $\mathcal{M}$ of multipliers M such that $M^*\phi$ is also positive. As an example, the original paper (Zames & Falb, 1968) was focused in preserving positivity for bounded and monotone nonlinearities; this class of multipliers is known as the Zames–Falb multipliers. Then if there exists a multiplier M within this class such that MG is strictly positive, then the linear system G in a feedback interconnection with any of the nonlinearities within the class (Fig. 1) is stable.

Time-domain quadratic constraints have long been considered as a tool for absolute stability in the Russian Literature; see in particular the work of Yakubovich, e.g. Yakubovich (1967). Safonov (1980) generalizes Zames' conic relation stability theorem (Zames, 1966a) to an absolute criterion based on topological separation. In Megretski and Rantzer (1997), a theorem based on IQCs in the frequency domain, which may be interpreted as a special case of these general theories, has been presented. It provides a unifying framework to combine nonlinearities using their classical multipliers, and conditions which can be easily tested in a linear matrix inequality (LMI) framework.

By contrast with the passivity theorem, the IQC theorem (Megretski & Rantzer, 1997) is derived using a homotopy argument where causality is not required. As a result, in IQC theory any multiplier preserving positivity for ϕ can be used and a canonical factorization is no longer required. This is sometimes stated as a distinguishing advantage of the IQC formulation (Jönsson, 1997; Megretski & Rantzer, 1997). But to date no significantly wider class of multipliers or improved stability results has yet been found that exploits this feature. This suggests the question: is the existence of a canonical factorization a necessary feature of multipliers for standard nonlinearity classes? In addition some authors still use the classical multiplier approach (Kulkarni, Pao, & Safonov, 2011); are their results conservative because they must then impose the canonical factorization?

Recently, a few papers have examined the connection between dissipativity and IQC theory (Materassi & Salapaka, 2009; Seiler, Packard, & Balas, 2010). In this paper we restrict our attention to the use of multipliers in the classical sense. In Fu, Dasgupta, and Soh (2005) a different factorization is analyzed, where M_+ and M_- are allowed to be "tall"; the use of this factorization does not demonstrate equivalency, since passivity theory requires invertible multipliers.

This paper focuses on these two questions. The main result is that under mild assumptions on the multiplier (rational, bounded, and positive) then both approaches lead to the same result. Moreover, it will be shown that the assumption on the positiveness of the multiplier does not affect the generality of the result if the class of nonlinearities Φ includes kI (henceforward, a scaled identity) where k is a positive constant. In particular, any LTI multiplier that preserves positivity must have a canonical factorization, except for certain pathological cases.

2. Problem definition

In this section some background concepts are summarized. The first subsection gives the notation and definitions that will be used throughout the paper. The second subsection introduces the canonical factorization and the condition for its existence. After that, the passivity theorem and its extension using multipliers are shown. Finally, the general IQC theorem is given. We assume the systems under consideration to be square. We make certain further restrictions on both the IQC framework and the passivity approach such that a straightforward comparison is possible.

2.1. Notation and definitions

The notation used throughout this paper is summarized in Table 1.

Let $\mathcal{L}_2^m[0,\infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f\colon[0,\infty)\to\mathbb{R}^m$. A truncation of the function f at T is given by $f_T(t)=f(t), \forall t\le T$ and $f_T(t)=0, \forall t>T$. In addition, f belongs to the extended space $\mathcal{L}_{2e}^m$ if $f_T\in\mathcal{L}_2^m$ for all $T>0$.

Let the system S be a map from $\mathcal{L}_{2e}^m[0,\infty)$ to $\mathcal{L}_{2e}^m[0,\infty)$, with input u and output Su. It is passive if $\langle u_T, Su_T\rangle\ge 0$ for all $T>0$ and $u\in\mathcal{L}_{2e}^m[0,\infty)$. It is (strictly) positive if $\langle u, Su\rangle(>)\ge 0$ for all $u\in\mathcal{L}_2^m[0,\infty)$. The system S is causal if $Su(t)=S(u_T)(t)$ for all $t<T$. Moreover, the system S is stable if for any $u\in\mathcal{L}_2^m[0,\infty)$, then $Su\in\mathcal{L}_2^m[0,\infty)$. The system S is bounded if there exists a constant γ such that $\|Su\|_2\le\gamma\|u\|_2$.

This definition of a positive system is standard, but it is not equivalent to the standard definition of a positive real system (Anderson & Vongpanitlerd, 1973), where causality is required. Although passivity and positivity definitions are often considered equivalent, the equivalence only holds for causal systems. Moreover, because passivity theory requires a inner product between the input and output, the space of the input should be the dual space of the space of the output; therefore, this paper is restricted to square systems.

Lemma 2.1 (*Section VI.9.1 in Desoer and Vidyasagar (1975)*). *Let $S:\mathcal{L}_{2e}^m[0,\infty)\to\mathcal{L}_{2e}^m[0,\infty)$ be a causal system. Then the system is passive if and only if it is positive.*

This paper focuses the stability of the feedback interconnection of a stable LTI system G and a bounded system ϕ, represented in Fig. 1 and given by

$$\begin{cases} v = f + Gw, \\ w = \phi v. \end{cases} \tag{2}$$

Since G is a stable LTI system, the exogenous input in this part of the loop can be taken as zero signal without loss of generality.