Automatica 48 (2012) 964–969

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Discrete-time switching linear system with constraints: Characterization and computation of invariant sets under dwell-time consideration ☆

Masood Dehghan, Chong-Jin Ong [1]

Department of Mechanical Engineering, National University of Singapore, 117576, Singapore

ARTICLE INFO

Article history:
Received 9 February 2011
Received in revised form
10 August 2011
Accepted 11 October 2011
Available online 21 March 2012

Keywords:
Switching systems
Minimal dwell-time
Constrained Systems
Set invariance
Stability

ABSTRACT

This paper introduces the concepts of Dwell-Time invariant/contractive (DT-invariant/contractive) set, Constraint Admissible DT-invariant/contractive (CADT-invariant/contractive) set for a discrete-time switching system under dwell-time switching. Main contributions of this paper include a characterization for a DT-contractive set, an algorithm for the computation of the maximal CADT-invariant set, a necessary and sufficient condition for asymptotic stability of the origin of switching systems under dwell-time switching and computation of the minimal dwell-time needed for asymptotic stability of the origin.

1. Introduction

This paper considers the following constrained discrete-time switching linear system:

$$x(t+1) = A_{\sigma(t)} x(t), \tag{1a}$$

$$x(t) \in \mathcal{X}, \quad \forall t \in \mathbb{Z}^+ \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \ldots, N\}$ is a time-dependent switching signal that indicates the current active mode of the system among N possible modes in $\mathcal{A} := \{A_1, \ldots, A_N\}$. The constraint set $\mathcal{X} \subset \mathbb{R}^n$ models physical state constraints imposed on the system, including those arising from the actuator via some appropriate state feedback if (1) is seen as a feedback system.

The study of such a system is quite active in the past decade. Most of the literature (Daafouz, Riedinger, & Iung, 2002; Liberzon & Morse, 1999) is concern with conditions that ensure stability of the system when $\sigma(\cdot)$ is an arbitrary switching function while others (Hespanha & Morse, 1999; Zhai, Hu, Yasuda, & Miche, 2002) consider designing the appropriate switching functions that ensure stability. With a few notable exceptions (Blanchini & Miani, 2007; Blanchini, Miani, & Savorgnan, 2007) the past literature does not consider the presence of constraints. When constraints are present, one major focus of research is the characterization of invariant/contractive sets that are constraint admissible. The existence of such invariant sets for system (1) is predicated on it being stable. Hence, studies of such sets often assume that $A_i, i \in \mathcal{I}_N$ is stable, which is a necessary condition for the stability of the origin of (1) under arbitrary switching. Additional conditions are required. The most common of these are those based on Lyapunov function consideration. For example, the origin of system (1) is stable under arbitrary switching upon the existence of a common quadratic Lyapunov function (Liberzon & Morse, 1999), (pairwise) switched Lyapunov functions (Daafouz et al., 2002), multiple Lyapunov functions (Branicky, 1998), composite quadratic functions (Hu, Ma, & Lin, 2008) or polyhedral Lyapunov functions (Blanchini et al., 2007). Another condition for stability is that based on dwell-time consideration. When all A_i is stable, stability of the origin can be ensured if the time duration spent in each subsystem is sufficiently long (Liberzon & Morse, 1999; Zhai et al., 2002). Upper bounds of the minimal dwell-time needed have also appeared (Blanchini, Casagrande, & Miani, 2010; Blanchini & Colaneri, 2010; Chesi, Colaneri, Geromel, Middleton, & Shorten, 2010; Geromel & Colaneri, 2006; Zhai et al., 2002).

This work is concern with the characterization and computation of invariant sets or contractive sets for system (1) when $\sigma(\cdot)$ is an admissible switching function that respects the dwell-time consideration. In the limiting case where the dwell-time is one sample period, $\sigma(\cdot)$ becomes an arbitrary switching

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Fen Wu under the direction of Editor Roberto Tempo (System and Control Theory).

E-mail addresses: masood@nus.edu.sg (M. Dehghan), mpeongcj@nus.edu.sg (C.-J. Ong).

[1] Tel.: +65 6874 2217; fax: +65 6779 1459.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.02.045

function, and the corresponding invariant/contractive sets and their computations have appeared in the literature (Blanchini & Miani, 2007). Hence, this work can also be seen as a generalization of those obtained for arbitrary switching systems. Other contributions of this work include: connection between stability of dwell-time switching systems and the stability of an associated arbitrary switching system, a necessary and sufficient stability condition for dwell-time switching systems, and a procedure that determines the minimal dwell-time needed for stability of the origin of system (1).

The rest of this paper is organized as follows. This section ends with a description of the notations used. Section 2 reviews some standard terminology and results for switching systems. Section 3 shows the main results on the characterization of the contractive set for system (1), its properties and computations, and a procedure for determination of the minimal dwell-time needed. Sections 4 and 5 contain, respectively, numerical examples and conclusions. All proofs of theorems except those needed in subsequent exposition are given in the Appendix.

The following standard notations are used. $\mathbb{Z}^+$ is the set of non-negative integers. Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$, A_j and b_k are the corresponding j-th row and the k-th element respectively while $\rho(A)$ denotes spectral radius of A. The floor function, $\lfloor a \rfloor$, is the largest integer that is less than scalar a. Positive definite (semi-definite) matrix, $P \in \mathbb{R}^{n\times n}$, is indicated by $P \succ 0 (\succeq 0)$ and I_n is the $n \times n$ identity matrix. Given a $P \succ 0$, $\mathcal{E}(P) := \{x : x^T P x \le 1\}$. The p-norm of a vector or a matrix is $\|\cdot\|_p$, $p = 1, 2, \infty$ with $\|\cdot\|$ refers to the 2-norm. Suppose $\alpha > 0$, $\mathcal{X} \subset \mathbb{R}^n$ is a compact set that contains 0 in its interior, then $\alpha\mathcal{X} := \{\alpha x : x \in \mathcal{X}\}$. Boldface $\mathbf{1}$ indicates the vector of all 1s. Other notations are introduced when needed.

2. Preliminaries

This section begins with a review of the definitions of switching time, dwell-time and admissible switching sequence. A switching sequence of (1) is denoted by $\mathcal{S}_\tau(t) = \{\sigma(t-1), \dots, \sigma(1), \sigma(0)\}$ with $\sigma(\cdot) \in \mathcal{I}_N$. Suppose $t_{s_0}, t_{s_1}, \dots, t_{s_k}, \dots$ are the switching instants of (1) with $t_{s_0} = 0$ and $t_{s_k} < t_{s_{k+1}}$. This means that $\sigma(t_{s_k}) \neq \sigma(t_{s_{k+1}})$ and $\sigma(t_{s_k}) = \sigma(t_{s_k}+1) = \cdots = \sigma(t_{s_{k+1}}-1)$ for all $k \in \mathbb{Z}^+$.

Definition 1. An admissible switching sequence of system (1), $\mathcal{S}_\tau(t)$, with switching instants $t_{s_0}, t_{s_1}, \dots, t_{s_k}, \dots$ has a dwell-time of τ means that $t_{s_{k+1}} - t_{s_k} \ge \tau$ for all $k \in \mathbb{Z}^+$. In addition, suppose t_{last} is the last switching time for an admissible sequence $\mathcal{S}_\tau(t)$, then $t - t_{\text{last}} \ge \tau$.

Throughout this paper, system (1) is assumed to satisfy the following assumptions: (A1) The spectral radius of each individual subsystem A_i, $i \in \mathcal{I}_N$ is less than 1; (A2) The constraint set $\mathcal{X}$ is a polytope represented by $\mathcal{X} = \{x : Rx \le \mathbf{1}\}$ for some appropriate matrix $R \in \mathbb{R}^{q\times n}$; (A3) (A_i, R) is observable for at least one $A_i \in \mathcal{A}$.

Assumption (A1) defines the family of systems considered herewith. The polyhedral assumption of (A2) is made to facilitate numerical computations of the invariant/contractive set of (1). Assumption (A3) ensures the compactness of the sets. It applies to only one $i \in \mathcal{I}_N$ since the invariant/contractive set is applicable to *all* admissible sequences including one where $\sigma(k) = i$ for all $k \in Z^+$. Of course, if (A3) is not satisfied, system (1) can be reformulated to consider only the observable subsystem of A_i.

3. Main results

This section begins with several definitions needed to precisely state the contractive/invariant condition of a set for system (1) with dwell-time consideration. For notational convenience, $A_{\mathcal{S}_\tau(t)}$ refers to the product $\Pi_{r=0}^{t-1} A_{\sigma(r)}$ associated with sequence $\mathcal{S}_\tau(t) = \{\sigma(t-1), \dots, \sigma(0)\}$.

Definition 2. A set $\Omega \subset \mathbb{R}^n$ is said to be Dwell-Time invariant (DT-invariant) w.r.t. system (1a) with a dwell-time τ if $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)} x \in \Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t.

Definition 3. A set $\Omega \subset \mathbb{R}^n$ is said to be Dwell-Time contractive (DT-contractive) w.r.t. system (1a) with a dwell-time τ and a contractive factor $\lambda \in (0, 1)$ if $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)} x \in \lambda\Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t.

It is clear from the definitions above that DT-contractivity requires stronger conditions than DT-invariance. Indeed, the existence of a DT-contractive set Ω implies that the origin of system (1) is asymptotically stable. This can be seen by associating Ω as the level set of an appropriately-defined Lyapunov function. See Remark 3 for details.

Definition 4. A set $\Omega \subset \mathbb{R}^n$ is said to be Constraint Admissible DT-contractive/invariant (CADT-contractive/invariant) w.r.t. system (1) with dwell-time τ and factor λ if it is DT-contractive/invariant and $x(t) \in \mathcal{X}$ for all $t \in \mathbb{Z}^+$.

While stating the requirements for a set to be DT/CADT-contractive/invariant, the above definitions are of limited use since $A_{\mathcal{S}_\tau(t)} x \in \lambda\Omega$ has to be satisfied by an infinite number of admissible sequences for all t. The next theorem shows how this can be avoided.

Theorem 1. *Suppose* (A1) *is satisfied and let* $\mathcal{T} := \{\tau, \tau+1, \dots, 2\tau-1\}$. *Then, a set* $\Omega \subset \mathbb{R}^n$ *is DT-contractive (with contraction factor* $\lambda \in (0,1)$*) for system* (1a) *with dwell time* τ, *if and only if for every* $x \in \Omega$,

$$A_i^t x \in \lambda\Omega \quad \text{for all } t \in \mathcal{T} \text{ and for all } A_i \in \mathcal{A}. \tag{2}$$

Proof. ($\Rightarrow$): The solution of (1) under an admissible switching function at time t is $x(t) = A_{\mathcal{S}_\tau(t)} x_0$ where

$$A_{\mathcal{S}_\tau(t)} = \cdots A_{i_\ell}^{k_\ell} \cdots A_{i_1}^{k_1} A_{i_0}^{k_0} \tag{3}$$

for some appropriate switching sequence $\mathcal{S}_\tau(t) = \{i_\ell, \dots, i_\ell, \dots, i_0, \dots, i_0\}$ where $i_j \in \mathcal{I}_N$ and $k_j := t_{s_{j+1}} - t_{s_j}$, $j = 0, 1, \dots, \ell$ being the corresponding duration times in each mode. Due to the dwell-time requirement, each $k_j \ge \tau$. Without loss of generality, consider any of the A_i^k on the right hand side of (3). This term can be decomposed into a product of matrices involving A_i^τ and one matrix from $\{A_i^\tau, A_i^{\tau+1}, \dots, A_i^{2\tau-1}\}$. To see this, let $q = \lfloor \frac{k-\tau}{\tau} \rfloor$ and

$$A_i^k = \left(A_i^\tau\right)^q A_i^{k-q\tau}. \tag{4}$$

Here, the superscript $k - q\tau$ of the last term corresponds to the remainder of $k-\tau$ when divided by τ and hence, assumes a value from $\mathcal{T} = \{\tau, \dots, 2\tau-1\}$. Consider the rightmost term of (4). Since for every $x \in \Omega$, $A_i^t x \in \lambda\Omega$ for all $t \in \mathcal{T}$ and for all $A_i \in \mathcal{A}$, it follows that $A_{i_0}^{k_0-q_0\tau} x_0 \in \lambda\Omega$ for any $x_0 \in \Omega$. Similarly, $(A_{i_0}^\tau)^{q_0} A_{i_0}^{k_0-q_0\tau} x_0 \in \lambda^{q_0+1}\Omega \subseteq \lambda\Omega$ as $A_i^\tau \Omega \subseteq \lambda\Omega$ from (2). Repeating this process for the rest of the terms in (3) and for all admissible sequences completes the proof.

($\Leftarrow$) Suppose Ω is DT-contractive with contraction λ, but there exists a $t \in \mathcal{T}$ and some $A_i \in \mathcal{A}$ such that $A_i^t \Omega \not\subseteq \lambda\Omega$. The sequence $\mathcal{S}_\tau(t) := \{i, i, \dots, i\}$, which is admissible, violates the DT-contractivity of Ω. □

An example that illustrates the proof is in order. Consider $\mathcal{A} = \{A_1, A_2\}$, $\tau = 3$ and $x(27) = A_{\mathcal{S}_\tau(27)} x_0 = A_1^8 A_2^9 A_1^{10} x_0$. Using the procedure described in the proof above, $x(27) =$