Automatica 47 (2011) 761–768

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Model reduction of homogeneous-in-the-state bilinear systems with input constraints[☆]

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ARTICLE INFO

Article history:
Available online 23 February 2011

Keywords:
Model reduction
Bilinear systems
Constraints
Multi-input/multi-output systems
Dynamic models

ABSTRACT

Homogeneous-in-the-state bilinear systems, appended by an additive disturbance, appear both from the discretization of some partial differential equations and from the bilinearization of certain nonlinear systems. They often have large state vectors that can be cumbersome for simulation and control system design. Our aim is to define a method, invariant to time transformations, for finding a reduced-order model with similar disturbance–output characteristics to those of the plant for all admissible input sequences. The inputs considered satisfy simple upper and lower bound constraints, representing saturating actuators. The approximation is based on a model truncation approach and a condition for the existence of such an approximation is given in terms of the feasibility of a set of linear matrix inequalities. A novelty of our work is in the definition of a new Gramian for this class of systems. Explicit error bounds on the scheme are included. The paper concludes with a demonstration of the proposed approach to the model reduction of a solar collector plant.

1. Introduction

Bilinear systems are a special class of nonlinear systems that are linear in the input and linear in the state, but not jointly linear in both. They have received considerable attention because they offer something of a halfway house between linear and nonlinear models. A review of applications and properties can be found in Bruni, DiPillo, and Koch (1974) and Mohler (1973). We consider a class of such systems known as continuous-time, homogeneous-in-the-state bilinear systems (Bruni et al., 1974) in the presence of an additive disturbance that are multi-input multi-output (MIMO) and can be written in the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + Rw(t), \tag{1a}$$

$$y(t) = Cx(t), \quad x(0) = x_0, \tag{1b}$$

where the state $x : [0, \infty) \to \mathbb{R}^n$, output $y : [0, \infty) \to \mathbb{R}^p$, disturbance $w : [0, \infty) \to \mathbb{R}^r$, $A, N_i \in \mathbb{R}^{n\times n}$ for all $i \in \{1, \dots, m\}$, $C \in \mathbb{R}^{p\times n}$, $R \in \mathbb{R}^{n\times r}$ and control input $u_i \in \mathcal{U}$ for all $i \in \{1, \dots, m\}$. We consider the case where $\mathcal{U} := \{u : [0, \infty) \to \mathbb{R} \mid \sup_t |u(t)| \le 1\}$. This choice of $\mathcal{U}$ corresponds to systems where the inputs are independently constrained as a result of saturating actuators, for example. Note that simple upper and lower bound constraints can be rewritten in this form by a simple change of variable (see the example in Section 4). Interest in such a model stems from the need for control of large-scale, input constrained nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) + h(x(t))w(t), \tag{2a}$$

$$y(t) = Cx(t), \quad x(0) = x_0, \tag{2b}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^{n\times r}$, $u_i \in \mathcal{U}$ for all $i \in \{1, \dots, m\}$ and $w : [0, \infty) \to \mathbb{R}^r$. With a linearization near an equilibrium point or the Carleman bilinearization, nonlinear systems of the form in (2) can sometimes be approximated by a system of the form in (1) (Bai & Skoogh, 2006). There is an additional requirement on g_i such that its linearization leads to a term strictly linear in x as opposed to an affine one. Examples of such systems can be found in a range of fields including ecological systems (Mohler, 1973), fluid mixing applications (Mathew, Mezić, Grivopoulos, Vaidya, & Petzold, 2007) and solar energy plants (Tenny, Rawlings, & Wright, 2004). They may at first glance seem to be a peculiar class of systems as they have a special property: once at the origin, the input cannot affect the system in the absence of a disturbance. Consider a white liquid with a disturbance representing the addition of a miscible red dye at a specific part of the domain. An input can stir the

[☆] The material in this paper was partially presented at the 2010 American Control Conference, June 30–July 2, 2010, Baltimore, MD, USA. This paper was recommended for publication in revised form by Associate Editor Graziano Chesi under the direction of Editor Roberto Tempo.

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doi:10.1016/j.automatica.2011.01.030

dye and the liquid to form a pink color, but once the resultant liquid is pink, no amount of stirring can separate the red and white components.

Control system design for constrained, nonlinear systems is notoriously difficult. Receding horizon approaches offer one option, although they require the recursive, online solution of open-loop optimal control problems (Tenny et al., 2004). For even moderately sized systems, this can be prohibitively expensive. Generating reduced-order models for use in simulation or control of such large-scale systems has therefore been the focus of considerable research. Some work has already been carried out on model reduction of general nonlinear systems (Scherpen, 1993) and bilinear systems (Hartmann, Zueva, & Schafer-Bung, submitted for publication; Hsu, Desai, & Crawley, 1983; Zhang & Lam, 2002) and so are applicable to the homogeneous-in-the-state bilinear case. In Scherpen (1993), observability and controllability functions are defined for a general nonlinear system. To compute such quantities, the solution to a Hamilton–Jacobi–Bellman type partial differential equation must be found—something notoriously difficult to do even for small systems. Similarly, there is no reason the reduced-order model would be bilinear (Gray & Mesko, 1998). The application of the approach from Scherpen (1993) to singularly perturbed bilinear systems is considered in Djennoune and Bettayeb (2005). The authors use the structure provided by a model containing simultaneously slow and fast dynamics to simplify the computation of the bilinear system Gramians although this is more a study into the structure of bilinear system functions than a description of a model reduction scheme. In Hartmann et al. (submitted for publication); Hsu et al. (1983) and Zhang and Lam (2002) the observability and controllability Gramians as defined by d'Alessandro, Isidori, and Ruberti (1974) form the basis of the reduction scheme, but differ in the fact that Hartmann et al. (submitted for publication) and Hsu et al. (1983) consider balanced truncation type approaches for model reduction, while Zhang and Lam (2002) consider a frequency domain $\mathcal{H}_2$ based reduction method. The Gramians in d'Alessandro et al. (1974) are defined in terms of the kernels of the Volterra series expansion of the state. A benefit of such an approach is that they can be computed as the solution of Generalized Lyapunov equations (Al-Baiyat & Bettayeb, 1993) of the form

$$A'Q + QA + \sum_{i=1}^{m} N_i' Q N_i + C'C = 0, \tag{3}$$

which for certain types of system can be computed efficiently (Al-Baiyat & Bettayeb, 1993). An interesting discussion of the physical interpretation of such a Gramian can also be found in Gray and Mesko (1998). In Hsu et al. (1983) the reduction is carried out by approximating the bilinear observability and controllability Gramians with lower-order ones via principal component analysis (Moore, 1981). The balanced truncation method can be shown to perform well on numerical examples (Hsu et al., 1983). However, there are certain drawbacks not highlighted. These are summarized in Section 3.1 and helps motivate the new results in this paper. In order to address the problems described, we consider two new Gramians, coined the D-Gramian and E-Gramian that have simple energy-based interpretations. We show that suitable examples of such Gramians can be computed as the solution of an LMI constrained optimization problem.

The main contribution of this paper is the definition of a new method for the model reduction of an homogeneous-in-the-state bilinear system. This involves the discussion of some problems with an existing approach, the definition of two new Gramians, and the introduction to a reduction scheme. The new reduction scheme is invariant to model time transformations and so, unlike the existing approaches, shows no degradation in performance when the units of time used in the plant modeling are changed. This issue is discussed in detail and demonstrated on a study of a model of a solar collector plant with the new scheme outperforming an existing method for reduced-order models of the same size. As a final contribution, a reduced-order model similar to the solar collector plant is shown to exist, thereby motivating future work into efficient control system design for such a plant.

This paper is organized as follows. The problem is formulated in Section 2. Sections 3.1 and 3.2 discuss two different definitions of Gramians about which a model reduction scheme can be based. The first is the one used in Hsu et al. (1983), the second a new construction and the paper's first contribution. In Section 3.3, a new reduction scheme is proposed and important properties discussed. Section 4 demonstrates the algorithm on a real-world example. Finally, some conclusions are given and future work proposed.

2. Formal problem statement

This work is focused on computing a reduced-order model with similar disturbance–output properties to those of the plant. To achieve similar disturbance–output behavior we consider minimizing the maximum $\mathcal{L}_2$-gain of the disturbance–error system for all feasible input sequences. A realization $(A, N_1, \ldots, N_m, C, R)$ refers to the system (1). The problem is formally written:

Sub-optimal disturbed model truncation: Given a $\gamma > 0$, find a projection matrix $T \in \mathbb{R}^{q\times n}$, $q < n$ satisfying:

$$\max_{u\in\mathcal{U}^m} \max_{\substack{w\in\mathcal{L}_2[0,\infty) \\ w\neq 0}} \frac{\int_0^\infty (y(t)-\hat{y}(t))'(y(t)-\hat{y}(t))\mathrm{d}t}{\int_0^\infty w'(t)w(t)\mathrm{d}t} \leq \gamma,$$

with $x_0 = 0$, where y is the output of the realization $(A, N_1, \ldots, N_m, C, R)$, $\hat{y}$ is the output of the realization $(TAT^+, TN_1T^+, \ldots, TN_mT^+, CT^+, TR)$ and u, w are the inputs and disturbances, respectively, applied to both systems. $\mathcal{L}_2[0, \infty)$ is the spaces of square integrable and Lebesgue measurable functions defined on the interval $[0, \infty)$, and T^+ represents the Moore–Penrose pseudo-inverse of T (Hogben, 2007).

Note that the matrix T projects the dynamics onto a subspace of $\mathbb{R}^n$, and so if $\hat{x}$ is the state of the reduced realization, then $T^+\hat{x}(t)$ is an approximation to $x(t)$. It should also be noted at this point that solutions to the sub-optimal disturbed model truncation problem are generally not unique, because realizations are typically invariant to a state coordinate transformation (Dullerud & Paganini, 2000). Also note that this is a sub-optimal model reduction scheme because we are constraining ourselves to the problem of finding a truncation matrix capable of achieving the desired performance.

Section 3 will show that the model reduction procedure to solve the sub-optimal disturbed model truncation problem is:

(1) Find a D-Gramian P and an E-Gramian Q using Algorithm 2 from Section 3.4;
(2) Find the balancing transformation U and transformed Gramians $\tilde{P} = \tilde{Q} = \Sigma$ using Algorithm 1 from Section 3.3;
(3) Examine the singular values (diagonal components of Σ) and decide upon a q to achieve satisfactory performance (using the error bounds of Theorem 7), then compute $T := \begin{bmatrix} I_{q\times q} & 0_{q\times(n-q)} \end{bmatrix} U$;
(4) Compute the reduced disturbed realization $(TAT^+, TN_1T^+, \ldots, TN_mT^+, CT^+, TR)$.

3. Solution

The solution approach involves using results from principal component analysis (PCA) (Jolliffe, 2002) to approximate a