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Finite-time stability theorem of stochastic nonlinear systems[☆]

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ABSTRACT

A new concept of finite-time stability, called stochastically finite-time attractiveness, is defined for a class of stochastic nonlinear systems described by the Itô differential equation. The settling time function is a stochastic variable and its expectation is finite. A theorem and a corollary are given to verify the finite-time attractiveness of stochastic systems based on Lyapunov functions. Two simulation examples are provided to illustrate the applications of the theorem and the corollary established in this paper.

1. Introduction

Up until now, much work has been done in the field of finite-time stability (FTS) and control. In the existing literature, finite-time stability of system equilibrium can be classified into two categories. One (Amato & Ariola, 2005; Amato, Ariola, & Cosentino, 2006; Amato, Ariola, & Dorato, 2001; Ambrosino, Calabrese, Cosentino, & Tommasi, 2009; Garcia, Tarbouriech, & Bernussou, 2009; Kushner, 1966; Lasarevic & Debeljkovic, 2005; Michel & Porter, 1972; Weiss & Infante, 1967; Yang, Li, & Chen, 2009; Zhang & An, 2008) is defined as follows: given a bound on the initial condition, the system state does not exceed a certain threshold during a specified time interval. The other (Nersesov & Perruquetti, 2008; Ryan, 1979) is defined thus: the system state reaches the system equilibrium in a finite time. To avoid confusion, the former is called finite-time boundedness, and the latter is called finite-time attractiveness.

This paper will mainly deal with finite-time attractiveness. Initial work on finite-time attractiveness mainly focused on discontinuous dynamic systems which can deteriorate the system transient performance (Ryan, 1979, 1991). In Bhat and Bernstein (1998) and Haimo (1986), the authors gave several conditions of finite-time attractiveness for first-order or second-order continuous autonomous systems. For multidimensional continuous autonomous systems, Bhat and Bernstein (2000) provided a sufficient and necessary condition for finite-time attractiveness involving the continuity of the settling time function at the origin. The other sufficient and necessary condition was given in Moulay and Perruquetti (2006) without assuming the continuity of the settling time function at the origin. Finite-time attractiveness was analyzed based on vector Lyapunov functions in Nersesov, Haddad, and Hui (2008) and Nersesov, Nataraj, and Avis (2009). Recently, finite-time attractiveness has been further extended to non-autonomous systems (Moulay & Perruquetti, 2008), switched systems (Orlov, 2005), time-delay systems (Moulay, Dambrine, Yeganefar, & Perruquetti, 2008), and impulsive dynamical systems (Nersesov & Perruquetti, 2008). It should be emphasized that the existing finite-time attractiveness literature considers only deterministic systems. After the success of finite-time attractiveness theory and its applications for deterministic systems, how to extend them to the case of stochastic systems, naturally became an important research area. Unfortunately, to the authors' knowledge, no work on finite-time attractiveness of stochastic systems has been done at the present stage.

The main goal of this paper is to fill the above gap, i.e., to extend the partial results of existing finite-time attractiveness for deterministic systems to the stochastic setting. First, we define a stochastic settling time function which is not only a function depending on the initial condition, but also a stochastic variable.

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Then, we further give the new concept of stochastically finite-time attractiveness. That is, the system origin is said to be finite-time attractive if the expectation of the stochastic settling time function of solutions starting from a neighborhood of the origin is finite. Based on the definition above, we establish a theorem and a corollary to verify the stochastically finite-time attractiveness, which are extensions of the existing results of finite-time attractiveness for deterministic systems.

2. Stochastic stability and stochastically finite-time attractiveness

Throughout this paper, $R_+ = [0, +\infty)$, R^n denotes the real n-dimensional space. For a given vector or matrix X, X^{T} denotes its transpose, $\mathrm{Tr}\{X\}$ denotes its trace when X is square, and $\|X\|$ denotes the Euclidean norm of a vector X. C^i denotes the set of functions with continuous ith partial derivatives. $E(x)$ denotes the expectation of stochastic variable x. $\mathcal{K}$ denotes the set of all functions, $R_+ \to R_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded.

2.1. Stochastic stability

Consider the following stochastic nonlinear system

$$\mathrm{d}x = f(x)\mathrm{d}t + g(x)\mathrm{d}w \tag{1}$$

where $x \in R^n$ is the system state vector, w is an r-dimensional independent standard Wiener process, and $f : R^n \to R^n$ and $g : R^n \to R^{n\times r}$ are continuous and satisfy $f(0) = 0, g(0) = 0$. In this paper, for simplicity, without loss of generality, we use 0 and x_0 to denote the initial time and the initial state of the system. The solution of system (1) with the initial state x_0 is denoted by $x(t, x_0)$.

Definition 1 (*Deng, Krstic, & Williams, 2001*)**.** The equilibrium $x = 0$ of system (1) is

- globally stable in probability if $\forall\epsilon > 0$ there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that
$$P\{\|x(t, x_0)\| < \gamma(\|x_0\|)\} \ge 1 - \epsilon, \quad \forall, t \ge 0,\ \forall x_0 \in R^n, \tag{2}$$
- globally asymptotically stable in probability if $x = 0$ is globally stable in probability, and
$$P\left\{\lim_{t\to\infty} \|x(t, x_0)\| = 0\right\} = 1, \quad \forall x_0 \in R^n. \tag{3}$$

For any given $V(x) \in C^2$, associated with stochastic system (1), the infinitesimal generator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\left\{g^{\mathrm{T}}(x)\frac{\partial^2 V}{\partial x^2} g(x)\right\}.$$

Lemma 1 (*Deng et al., 2001*)**.** *Consider system (1) and suppose there exist a C^2 function $V : R^n \to R_+$ and class $\mathcal{K}_\infty$ functions α_1, α_2 and α_3 such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \tag{4}$$

$$\mathcal{L}V(x) \le -\alpha_3(\|x\|). \tag{5}$$

Then the equilibrium $x = 0$ is globally asymptotically stable in probability.

2.2. Definition of stochastic finite-time attractiveness

Definition 2 (*Stochastic Settling Time Function*)**.** For system (1), define $T_0(x_0, w) = \inf\{T \ge 0 : x(t, x_0) = 0, \forall t \ge T\}$, which is called the stochastic settling time function.

Remark 1. If $T_0(x_0, w) \le +\infty$, we say that $T_0(x_0, w)$ exists. Obviously, the stochastic settling time function does not always exist for stochastic system (1). However, if stochastic system (1) is asymptotically stable in probability, then according to Eq. (3) in Definition 1, the solution $x(t, x_0)$ of system (1) will converge to zero in a finite or infinite time with probability one, which implies that the stochastic settling time $T_0(x_0, w)$ exists with probability one. It must be emphasized that $T_0(x_0, w)$ is not only a function of x_0, but also a stochastic variable for a fixed x_0, which is quite different from the case of deterministic systems (Bhat & Bernstein, 1998, 2000; Haimo, 1986; Moulay et al., 2008; Moulay & Perruquetti, 2006, 2008; Nersesov et al., 2008, 2009; Nersesov & Perruquetti, 2008; Orlov, 2005; Ryan, 1979, 1991).

Definition 3 (*Stochastically Finite-Time Attractiveness*)**.** For stochastic system (1), the origin $x = 0$ is said to be globally stochastically finite-time attractive, if for $x_0 \in R^n$, the following conditions hold.

(i) Stochastic settling time function $T_0(x_0, w)$ exists with probability one.
(ii) Provided that $T_0(x_0, w)$ exists, then $E[T_0(x_0, w)] < \infty$.

Remark 2. The finite-time attractiveness of stochastic systems is defined in a more difficult way than that of deterministic systems (Bhat & Bernstein, 1998, 2000; Haimo, 1986; Moulay et al., 2008; Moulay & Perruquetti, 2006, 2008; Nersesov et al., 2008, 2009; Nersesov & Perruquetti, 2008; Orlov, 2005; Ryan, 1979, 1991) owing to the stochastic property. Condition (i) guarantees the existence of $T_0(x_0, w)$ almost surely. Only under condition (i), one can further discuss condition (ii). From condition (ii), it can be seen that the finite-time property is evaluated by the expectation of $T_0(x_0, w)$.

2.3. Finite-time stability theorem using the Lyapunov function

Lemma 2. *Assume that $\alpha(\cdot) : R \to R$ and $\beta(\cdot) : R^n \to R$ are two smooth functions, and $x(t)$ is the solution of system (1). Then for any $a \le b$, the following equation holds*

$$\int_a^b \frac{\mathrm{d}\alpha(\beta(x(t)))}{\mathrm{d}\beta}\mathrm{d}\beta(x(t)) = \alpha(\beta(x(t)))\Big|_a^b - \frac{1}{2}\int_a^b \frac{\mathrm{d}^2\alpha}{\mathrm{d}\beta^2}\mathrm{Tr}\left\{\left(\frac{\partial\beta}{\partial x}g\right)^{\mathrm{T}}\frac{\partial\beta}{\partial x}g\right\}\mathrm{d}t \tag{6}$$

where $\alpha(\beta(x(t)))\,|_a^b = \alpha(\beta(x(b))) - \alpha(\beta(x(a)))$.

Proof. According to the Itô formula, one has

$$\begin{aligned}\mathrm{d}[\alpha(\beta(x))] &= \left[\frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\frac{\partial\beta}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\left\{g^{\mathrm{T}}\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}\beta^2}\left(\frac{\partial\beta}{\partial x}\right)^{\mathrm{T}}\frac{\partial\beta}{\partial x} + \frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\frac{\partial^2\beta}{\partial x^2}\right)g\right\}\right]\mathrm{d}t + \frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\frac{\partial\beta}{\partial x}g(x)\mathrm{d}w \\ &= \frac{\mathrm{d}\alpha}{\mathrm{d}\beta}d[\beta(x)] + \frac{1}{2}\frac{\mathrm{d}^2\alpha}{\mathrm{d}\beta^2}\mathrm{Tr}\left\{\left(\frac{\partial\beta}{\partial x}g\right)^{\mathrm{T}}\frac{\partial\beta}{\partial x}g\right\}\mathrm{d}t.\end{aligned} \tag{7}$$

Integrating both sides from a to b leads to (6). This completes the proof. □

Theorem 1. *Consider system (1). If there exist a positive definite, twice continuous differentiable and radially unbounded Lyapunov*