Automatica 45 (2009) 1601–1610

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Box invariance in biologically-inspired dynamical systems☆

Alessandro Abate[a,*], Ashish Tiwari[b], Shankar Sastry[c]

[a] Department of Aeronautics and Astronautics, Stanford University, CA, United States
[b] Computer Science Laboratory, SRI International, Menlo Park, CA, United States
[c] Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA, United States

ARTICLE INFO

Article history:
Received 3 April 2008
Received in revised form
16 February 2009
Accepted 26 February 2009
Available online 29 March 2009

Keywords:
Positive invariance
Invariant sets
Biological systems
Nonquadratic Lyapunov functions
Switched and Hybrid systems

ABSTRACT

A dynamical system is box invariant if there exists a box-shaped positively invariant region. We show that box invariance can be checked in cubic time for linear and affine systems, and that it remains decidable for classes of nonlinear systems of interest (with polynomial structure). We present results on the robustness of box invariance for linear systems using spectral properties of Metzler matrices. We also present sufficient conditions for establishing box invariance of switched and hybrid systems. In general, we argue that box invariance is a characteristic of many biologically-inspired dynamical models.

1. Introduction

An *invariant* set is a subset of the state space of a dynamical system with the property that, if the system state is in this set at some time, then it will stay in the set indefinitely in the future (Blanchini, 1999). An invariant set is extremely useful from the perspective of formal analysis and verification (Clarke, Grumberg, & Peled, 2000). The task in formal verification is to show that none of the trajectories of a given dynamical system violate a given property, such as a liveness or safety property, or in the opposite instance to find "witnesses" that do not abide by such properties. *Safety* specifications form an important class of properties, which encode the condition that a system can never reach a given subset of "unsafe" or "bad" states. Direct verification of safety properties is difficult because computing the set of reachable states is often infeasible. However, an invariant set can be used to verify a safety property by showing that it encloses all reachable states, but none of the unsafe states. From a dual perspective, invariants can be used to look at *reachability* properties, where the objective is to verify if any trajectory of the system, starting from a region of the state space, will reach a target set (which is again a subset of the state space). The concept of invariance can also be related to certain notions of stability (Podelski & Wagner, 2006). This motivates the need to develop effective and constructive approaches to discover invariant sets for dynamical systems—and especially invariant sets with simple shapes.

Positively invariant sets can be obtained by exploiting the property that their boundaries may correspond to level surfaces of a proper Lyapunov-like function. This approach has been the source of several results on the existence of positively invariant sets (Blanchini, 1999; Kiendl, Adamy, & Stelzner, 1992). However, this is quite restrictive in general, since systems that are not stable can still have useful invariant sets.

In this paper, we focus on positively invariant sets that are in the form of a box, that is, a hyper-rectangular region specified by giving (upper and lower) bounds for each state variable. The concept of box invariance is related to a number of studies in the literature (Blanchini, 1999) (see Section 2.1). For instance, Kiendl et al. (1992) look at the use of vector norms to study stability. The notions that are developed in the present study are related to that of component-wise stability (Pastravanu & Voicu, 2003; Voicu, 1984), as well as to the concepts of practical stability and Lagrange stability (Passino, Burgess, & Michel, 1995).

☆ Research supported by the grants CCR-0225610 and DAAD19-03-1-0373, and in part by NSF CNS-0720721, and NASA NNX08AB95A. The material in this paper was partially presented at the 2nd ADHS in 2006, and at the 46th CDC in 2007. This paper was recommended for publication in revised form under the direction of Editor Andrew R. Teel.

* Corresponding address: Stanford University, Durand Building, Room 250496 Lomita Mall 94305-4035 Stanford, United States. Tel.: +1 415 225 2778; fax: +1 650 723 3738.

E-mail addresses: aabate@stanford.edu (A. Abate), tiwari@csl.sri.com (A. Tiwari), sastry@eecs.berkeley.edu (S. Sastry).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.02.028

The study of several systems, especially models drawn from the domain of systems biology, has suggested that they frequently admit box-shaped, positively invariant sets. This seems natural in retrospect since state variables often correspond to physical quantities that are naturally constrained and tend to either degrade, or remain conserved. In this paper, we are interested in the practical aspects of the notion of box invariance. In particular, we focus on how complex it is to check for box invariance of a dynamical model, as well as to construct a particular box, whenever possible. More precisely, we show that it is computationally feasible to check if a dynamical system is invariant with respect to a box set, and to explicitly find out box invariant sets for a large class of dynamical systems (in particular, biological ones). Because of the discussed connections with other notions in systems theory, it is then argued that box invariance is an ideal concept for building analysis and verification tools to investigate such systems.

Outline. We formally define the notion of box invariance in Section 2. Next, we present necessary and/or sufficient characterizations of this notion for linear (Section 3), affine (Section 3.3), and classes of nonlinear systems (Section 4) that are especially meaningful for models of biological systems. Box invariance of linear systems is strongly related to the theory of Metzler matrices, as explained in Section 3.1. Using this connection, we perform robustness analysis of box invariant systems in Section 3.2. In Section 5, we extend the study to the more general case of switched and hybrid systems. All throughout, we will present computational complexity results and illustrate the introduced concepts using examples from systems biology.

2. The concept of box invariance

We consider general and uncontrolled dynamical systems of the form $\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$. We assume the basic boundedness and Lipschitz properties that ensure the existence of a unique solution of the vector field, given any possible initial condition. A rectangular box around a point $\boldsymbol{x}_0$ is specified using two diagonally opposite points $\boldsymbol{l}$ and $\boldsymbol{u}$, where $\boldsymbol{l} < \boldsymbol{x}_0 < \boldsymbol{u}$ (interpreted component-wise) and is defined as $Box(\boldsymbol{l}, \boldsymbol{u}) := \{\boldsymbol{x} \mid \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$. Such a box has $2n$ faces consisting of n lower and n upper faces. The jth lower face is defined as $Face\,L^j(\boldsymbol{l}, \boldsymbol{u}) := \{\boldsymbol{x} \in Box(\boldsymbol{l}, \boldsymbol{u}) \mid x_j = l_j\}$ and the jth upper face is defined as $Face\,U^j(\boldsymbol{l}, \boldsymbol{u}) := \{\boldsymbol{x} \in Box(\boldsymbol{l}, \boldsymbol{u}) \mid x_j = u_j\}$, for $j \in \{1, \ldots, n\}$.

Definition 1 (*Box Invariant System*). A dynamical system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is said to be *box invariant* around an equilibrium point $\boldsymbol{x}_0$ if there exists a finite rectangular box $Box(\boldsymbol{l}, \boldsymbol{u})$ around $\boldsymbol{x}_0$ such that $f(\boldsymbol{y})_j \leq 0$ whenever $\boldsymbol{y} \in Face\,U^j(\boldsymbol{l}, \boldsymbol{u})$ and $f(\boldsymbol{y})_j \geq 0$ whenever $\boldsymbol{y} \in Face\,L^j(\boldsymbol{l}, \boldsymbol{u})$. The system is said to be *strictly box invariant* if the inequalities hold strictly.

An equivalent definition of box invariant system can be given as a system that admits a box as a positively invariant set. In the case of multiple equilibria, either finite or infinite in cardinality, we require the existence of (possibly different) boxes for each of them.

Note that the existence of a box is unaffected by the reordering of state variables and by rotations by multiples of $\pi/2$. It also displays invariance under independent stretches of the coordinates. Nevertheless, it is not invariant under general linear transformations.

Definition 2 (*Symmetrical Box Invariance*). A system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is said to be *symmetrically box invariant* around the equilibrium $\boldsymbol{x}_0$ if there exists a point $\boldsymbol{u} > \boldsymbol{x}_0$ (interpreted component-wise) such that the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is box invariant with respect to the box $Box(2\boldsymbol{x}_0 - \boldsymbol{u}, \boldsymbol{u})$.

2.1. Box invariance through vector norms

The boundary of a box can be seen as a level surface of a function defined by a vector norm. Let $\|\boldsymbol{x}\|_\infty = \max\{|x_i|, i = 1, \ldots, n\}$ denote the infinity norm on an n-dimensional Euclidean space. Let D be an $n \times n$ positive diagonal matrix. Any level set of the positive real-valued function $\|D\boldsymbol{x}\|_\infty$ coincides with a hyper-rectangle in $\mathbb{R}^n$ that is symmetric around the origin. Specifically, for any positive constant $c \in \mathbb{R}$, $\{\boldsymbol{x} \mid \|D\boldsymbol{x}\|_\infty \leq c\} = Box(-cD^{-1}\mathbf{1}, cD^{-1}\mathbf{1})$, where $\mathbf{1}$ is the n-dimensional unity vector. Accordingly, symmetrical box invariance has in part already, though not explicitly, been studied in the literature by exploring when $\|D\boldsymbol{x}\|_\infty$ is a Lyapunov function for a dynamical system (Pastravanu & Voicu, 2003; Voicu, 1984). For linear systems, a sufficient condition for this to hold is the existence of a matrix Q of proper size, with $\mu(Q) < 0$, such that $WA = QW$ (Kiendl et al., 1992). Here $\mu(Q)$ is a matrix measure defined as $\mu(Q) = \lim_{\Delta \to 0^+} \frac{\|I + \Delta Q\|_\infty - 1}{\Delta}$.

Whereas the existence of box invariants is closely related to Lyapunov stability under infinity vector norms for linear systems (see Theorems 2 and 3), this is not so for more general nonlinear and hybrid systems. Invariants are also not easy to compute in general. This motivates the search for invariants of a simple form, such as a box. As we show in the present work, box invariants can be easily *computed* using simple constraint-solving techniques.

3. Box invariant linear and affine systems

Given a linear system and a box around its equilibrium point, the problem of checking whether the system is box invariant with respect to the given box can be solved by verifying the related condition only at the 2^n vertices of the box (rather than on all the points of the surface of the box). The set of vertices, $Vert(\boldsymbol{l}, \boldsymbol{u})$, of the box $Box(\boldsymbol{l}, \boldsymbol{u})$ is defined as $Vert(\boldsymbol{l}, \boldsymbol{u}) = \{\boldsymbol{x} \mid x_i = l_i \vee x_i = u_i, \forall i\}$.

Proposition 1. *A linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}^n$, is box invariant if there exist two points $\boldsymbol{l} \in (\mathbb{R}^-)^n$ and $\boldsymbol{u} \in (\mathbb{R}^+)^n$ such that for each point $\boldsymbol{c} \in Vert(\boldsymbol{l}, \boldsymbol{u})$, we have $A\boldsymbol{c} \sim \mathbf{0}$, where $\sim_i$ is $\leq$ if $c_i = u_i$ and $\sim_i$ is $\geq$ if $c_i = l_i$.*

The proof follows the observation that the inequalities state that the vector field points inwards on the 2^n vertices in $Vert(\boldsymbol{l}, \boldsymbol{u})$, and that it is possible to extend by linearity the value of the vector field at other points on the faces of the box. Proposition 1 claims that box invariance of linear systems can be checked by testing the satisfiability of $n2^n$ linear inequality constraints, over $2n$ unknowns (given by $\boldsymbol{l}$ and $\boldsymbol{u}$). Lemmas 1 and 2 will allow us to simplify this requirement to testing n linear inequalities over n variables. Observe that the notion of box invariance and symmetrical box invariance are equivalent for linear systems:

Lemma 1. *A linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where $A \in \mathbb{R}^{n \times n}$, is box invariant if and only if it is symmetrically box invariant.*

Proof. If the linear system is symmetrically box invariant, then it is clearly also box invariant. To prove the converse, assume that the linear system is box invariant with respect to the box $Box(\boldsymbol{l}, \boldsymbol{u})$, where $\boldsymbol{l} \in (\mathbb{R}^-)^n$ and $\boldsymbol{u} \in (\mathbb{R}^+)^n$. We will show that the linear system is also box invariant with respect to the (symmetrical) box $Box(-\boldsymbol{c}, \boldsymbol{c})$, where $c_i = min(|l_i|, |u_i|)$. Consider first $i = 1$ and the case when $u_1 \leq -l_1$ so that $c_1 = u_1$. On the face $Face\,U^1(\boldsymbol{l}, \boldsymbol{u})$ of the $x_1 = u_1$ hyper-surface, by definition of $\boldsymbol{c}$, we have $Face\,U^1(-\boldsymbol{c}, \boldsymbol{c}) \subseteq Face\,U^1(\boldsymbol{l}, \boldsymbol{u})$. Hence, $(A\boldsymbol{x})_1 \leq 0, \forall \boldsymbol{x} \in Face\,U^1(-\boldsymbol{c}, \boldsymbol{c})$. Since $A(-\boldsymbol{x}) = -A\boldsymbol{x}$, we also get $(A\boldsymbol{x})_1 \geq 0$ for all $\boldsymbol{x} \in Face\,L^1(-\boldsymbol{c}, \boldsymbol{c})$. The opposite case when $-l_1 < u_1$ is similar. Repeating this argument for $i = 2, 3, \ldots, n$, completes the proof. □