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A generalized multi-period mean–variance portfolio optimization with Markov switching parameters ☆

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ABSTRACT

In this paper, we deal with a generalized multi-period mean–variance portfolio selection problem with market parameters subject to Markov random regime switchings. Problems of this kind have been recently considered in the literature for control over bankruptcy, for cases in which there are no jumps in market parameters (see [Zhu, S. S., Li, D., & Wang, S. Y. (2004). Risk control over bankruptcy in dynamic portfolio selection: A generalized mean variance formulation. *IEEE Transactions on Automatic Control, 49*, 447–457]). We present necessary and sufficient conditions for obtaining an optimal control policy for this Markovian generalized multi-period mean–variance problem, based on a set of interconnected Riccati difference equations, and on a set of other recursive equations. Some closed formulas are also derived for two special cases, extending some previous results in the literature. We apply the results to a numerical example with real data for risk control over bankruptcy in a dynamic portfolio selection problem with Markov jumps selection problem.

1. Introduction

The mean–variance portfolio selection problem was transformed with Markowitz's seminal work in Markowitz (1952). Since then, research on this subject has increased, in order to provide financial models with more realistic assumptions. Nowadays, there is extensive literature about this subject, with some extensions, as can be seen, for instance, in Costa and Nabholz (2002), Costa and Paiva (2002), Howe and Rustem (1997), Howe, Rustem, and Selby (1996), Roll (1992), Rustem, Becker, and Marty (1995) and Steinbach (2001), among others. One of the main advantages of the mean–variance criterion, is that it has a simple and clear interpretation in terms of individual portfolio choice and utility optimization, although some of its drawbacks are nowadays well known. In Li and Ng (2000), Li and Ng introduced a technique to tackle the multi-period mean–variance problem, with market uncertainties reproduced by stochastic models, in which the key parameters, expected return and volatility, are deterministic. This problem was also analyzed from a geometric point of view in Leippold, Trojani, and Vanini (2004), for the case with intermediate restrictions in Costa and Nabholz (2007), for the continuous-time case in Zhou and Li (2000), and for other related optimization problems in Chen, Li, and Zhou (1998), Dragan and Morozan (2004), Li and Zhou (2002), Li, Zhou, and Rami (2003), Lim and Zhou (1999) and Liu, Yin, and Zhou (2005). More recently there has been an increased interest in the study of financial models in which those key parameters are modulated by a Markov chain, see for instance Bäuerle and Rieder (2004), Çakmak and Özekici (2006), Yin and Zhou (2004), Zhang (2000) and Zhou and Yin (2003). Such models can better reflect the market environment, since the overall assets usually move according to a major trend given by the state of the underlying economy, or by the general mood of the investors.

The generalized multi-period mean–variance problem can be seen as an stochastic control problem, in which the objective function is formed by a weighted sum of a linear combination of the expected value, and square of the expected value of the wealth, and the expected value of the square of the wealth. As we are going to see next, a great variety of mean–variance models with intermediate restrictions and/or intermediate costs in the objective function can be derived from this generalized formulation. The usefulness of adopting this kind of criterion, is that in several situations, investor managers have to report their portfolio's return on a periodic basis to their beneficiaries, clients or to governmental authorities, so that intermediate performances are as important as the final one. Therefore more traditional mean–variance problems, which regards the performance only

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at the final value, would not be the most appropriate for these situations.

One example of the generalized multi-period mean–variance problem would be the case in which the linear combination reflects a trade-off between the expected value and the risk (variance) of the portfolio. For problems with constraints on the expected wealth and/or variance of the wealth, a primal–dual method is normally used, consisting of two optimization problems. For example, in Zhu, Li, and Wang (2004), the authors introduced a risk control over bankruptcy problem for dynamic portfolio selection. The basic idea is to control the probability of a portfolio falling below a specified level. Using the Tchebycheff inequality, the constraints are written in terms of the expected value and variance of the wealth. The first Lagrangian maximization problem, which is a multi-period generalized mean–variance problem, is formed by attaching to the objective function the constraints multiplied by the nonnegative Lagrangian multipliers. As remarked in Zhu et al. (2004), one key difficulty in solving this multi-period generalized mean–variance problem is the non-separability of the associated stochastic control problem, from the dynamic point of view. A solution procedure is presented in Zhu et al. (2004) based on an auxiliary problem, solvable by using dynamic programming, with a vector of auxiliary parameters, named λ. The solution of the first Lagrangian maximization problem is then achieved in Zhu et al. (2004) by setting the value of λ as the solution of a set of linear equations, if the inverse of an appropriate matrix exists or, otherwise, by a line search method. After this, the Lagrangian dual minimization problem must be solved over the Lagrangian multipliers (see Zhu et al. (2004)).

In general, mean–variance problems with restrictions would require a numerical procedure, as described above, to solve the dual minimization problem. However there are some special situations in which an exact solution can be derived analytically. The cases in which there is a restriction only on the final time T and the objective function considers only the final value of the variance or expected value of the wealth correspond to the traditional multi-period Markowitz's mean–variance selection problems. These problems were solved analytically in Li and Ng (2000) for the case with no Markovian jumps, and in Çakmak and Özekici (2006); Zhou and Yin (2003) for the Markovian jump case, with closed formulas for the optimal control strategy derived. The case in which there are intermediate restrictions but no jumps was analyzed in Costa and Nabholz (2007).

In this paper, we consider a multi-period generalized mean–variance model with Markov switching in the key market parameters. As in Zhu et al. (2004), we consider an auxiliary problem treatable from the dynamic point of view to analyze this problem, with an auxiliary vector of parameters λ. Our main result is to derive necessary and sufficient conditions for obtaining an optimal control policy for this multi-period Markovian generalized mean–variance problem, based on a set of interconnected Riccati difference equations, and some other recursive equations, which lead to recursive procedures for obtaining the desired solution. It is important to stress that previous papers on this subject (Çakmak & Özekici, 2006; Li & Ng, 2000; Zhu et al., 2004) obtained only necessary conditions for optimality of the control strategy. As far as the authors are aware, no sufficient condition had been obtained before. The closed solutions of two special cases are also provided, extending some previous results in the literature. When compared with the no jumps case, our expression for the auxiliary parameter λ is presented in a more explicit form than that in Zhu et al. (2004), providing a more direct way to compute the optimal control strategy for the multi-period generalized mean–variance problem. Moreover, we apply the obtained results to investigate a numerical example with real data for risk control over bankruptcy in a dynamic portfolio selection problem with Markov jumps selection problem.

This paper is organized as follows. In Section 2 we formulate the model and the problems to be investigated. In Section 3, an optimal control policy for an auxiliary problem, as well as the expected value and variance of the investor's wealth are analytically derived. Such a policy is obtained from the solution of a set of interconnected Riccati difference equations. Our main results are in Section 4, where we provide necessary and sufficient conditions for the solution of the generalized mean–variance problem, and a set of recursive equations, one set based on the necessary condition, and another set based on the sufficient condition, to derive the solution of the problem. The closed solution of two particular mean–variance problems are obtained in Section 5. A numerical simulation for the risk control over bankruptcy is investigated in Section 6. The paper is concluded in Section 7 with some final remarks.

2. Problem formulation

2.1. Definitions and the financial model

Throughout the paper we shall denote by $\mathbb{R}^n$ the n-dimensional Euclidean real space and by $\mathbb{R}^{n\times m}$ the Euclidean space of all $n \times m$ real matrices. For a sequence of numbers $a_1, \ldots, a_m$, we shall denote by $\mathrm{diag}(a_i)$ the diagonal matrix in $\mathbb{R}^{m\times m}$ formed by the element a_i at the ith diagonal, $i = 1, \ldots, m$. The superscript $'$ will denote the transpose of a vector or matrix. The variance of a random variable X will be denoted by $\mathrm{Var}(X)$.

We will consider a financial market with $n + 1$ risky securities on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. The assets' price will be described by the random vector $\bar{\mathrm{S}}(t) = (\mathrm{S}_0(t), \ldots, \mathrm{S}_n(t))'$ taking values in $\mathbb{R}^{n+1}$ with $t = 0, \ldots, T$. Set $\bar{\mathrm{R}}(t) = (\mathrm{R}_0(t), \ldots, \mathrm{R}_n(t))'$, with $\mathrm{R}_i(t) = \frac{\mathrm{S}_i(t+1)}{\mathrm{S}_i(t)}$. We assume that the random vector $\bar{\mathrm{R}}(t)$ satisfies the following equation:

$$\bar{\mathrm{R}}(t) = [\bar{e} + \bar{\mu}_{\theta(t)}(t)] + \bar{\sigma}_{\theta(t)}(t)\, W(t), \tag{1}$$

where $\bar{e} = \left(1, e'\right)'$, with $e \in \mathbb{R}^n$ a vector with $1's$ in all its components. Here $\{\theta(t); t = 0, \ldots, T\}$ is a finite-state discrete-time Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$, and $\{W(t); t = 0, \ldots, T\}$ is a sequence of $(n + 1)$-dimensional independent random vectors with zero mean and covariance I (identity matrix). We assume that $\{W(t), \theta(t)\}$ are mutually independent, and that the underlying Markov state $\{\theta(t)\}$ is observable. From (1), the expected return, variance and covariance of the assets at time t are affected by local or global factors, which are represented by the market operation mode $\theta(t) \in \mathcal{M}$. $\mathcal{P}$ is a probability measure such that $\mathcal{P}(\theta(t+1) = j | \theta(0), \ldots, \theta(t) = i) = \mathcal{P}(\theta(t+1) = j | \theta(t) = i) = p_{ij}(t), p_{ij}(t) \geq 0$ and $\sum_{j\in\mathcal{M}} p_{ij}(t) = 1$, for $t = 0, \ldots, T-1$ and $i, j \in \mathcal{M}$. We set for $t = 0, \ldots, T$, $P(t) = \left[p_{ij}(t)\right]_{m\times m} \in \mathbb{R}^{m\times m}$, $\pi_i(t) = \mathcal{P}(\theta(t) = i)$, $\pi(t) = (\pi_1(t), \ldots, \pi_m(t))'$. As pointed out in Yin and Zhou (2004), page 351, (1) can be seen as a frequently used discrete-time approximation of the geometric Brownian motion model of stock prices return in continuous-time. Although this approximation does not necessarily produce nonnegative stock prices, in practice, it is essentially the same as the discrete-time approximation obtained from the geometric formulation, which ensures non negativity of stock prices, see Yin and Zhou (2004).

As in Costa, Fragoso, and Marques (2005), for $z = (z_1, \ldots, z_m)' \in \mathbb{R}^m$, we define the operator $\mathcal{E}(z, t) = (\mathcal{E}_1(z, t), \ldots, \mathcal{E}_m(z, t))$ as $\mathcal{E}_i(z, t) = \sum_{j=1}^{m} p_{ij}(t)\, z_j$, for $i \in \mathcal{M}$. For notational simplicity, we shall omit from now on the variable t in $\mathcal{E}_i(z, t)$. The filtration $\mathcal{F}_t$ is such that the random vectors $\{\bar{\mathrm{S}}(k); k = 0, \ldots, t\}$ and Markov chain $\{\theta(k); k = 0, \ldots, t\}$ are $\mathcal{F}_t$-measurable.

When the market operation mode is $\theta(t) = i \in \mathcal{M}$, $\bar{\mu}_i(t) \in \mathbb{R}^{n+1}$ represents the vector with the expected returns of the assets,