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Approximately bisimilar symbolic models for nonlinear control systems☆

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ABSTRACT

Control systems are usually modeled by differential equations describing how physical phenomena can be influenced by certain control parameters or inputs. Although these models are very powerful when dealing with physical phenomena, they are less suited to describe software and hardware interfacing with the physical world. For this reason there is a growing interest in describing control systems through *symbolic models* that are abstract descriptions of the continuous dynamics, where each "symbol" corresponds to an "aggregate" of states in the continuous model. Since these symbolic models are of the same nature of the models used in computer science to describe software and hardware, they provide a unified language to study problems of control in which software and hardware interact with the physical world. Furthermore, the use of symbolic models enables one to leverage techniques from supervisory control and algorithms from game theory for controller synthesis purposes. In this paper we show that every incrementally globally asymptotically stable nonlinear control system is approximately equivalent (bisimilar) to a symbolic model. The approximation error is a design parameter in the construction of the symbolic model and can be rendered as small as desired. Furthermore, if the state space of the control system is bounded, the obtained symbolic model is finite. For digital control systems, and under the stronger assumption of incremental input-to-state stability, symbolic models can be constructed through a suitable quantization of the inputs.

1. Introduction

The idea of using models at different levels of abstraction has been successfully used in the formal methods community with the purpose of mitigating the complexity of software verification. A central notion when dealing with complexity reduction, is the one of bisimulation equivalence, introduced by Milner (1989) and Park (1981) in the 80s'. The key idea is to find and compute an equivalence relation on the state space of the system that respects the system dynamics. This equivalence relation induces a new system on the quotient space that shares most properties of interest with the original model. This approach leads to an alternative methodology for the analysis and control of large-scale control systems. In fact, from the analysis point of view, symbolic models provide a unified framework for describing continuous systems as well as hardware and software interacting with the physical environment. Furthermore, the use of symbolic models allows one to leverage the rich literature on supervisory control (Ramadge & Wonham, 1987) and algorithmic approaches to game theory (Arnold, Vincent, & Walukiewicz, 2003), for controller design.

After the pioneering work of Alur and Dill (1994) that showed existence of symbolic models for timed automata, researchers tried to identify more general classes of continuous systems admitting finite bisimulations. The existing results can be roughly classified into four main different lines of research:

(i) *Simulation/bisimulation*: symbolic models have been studied in Girard (2007), Tabuada (2007) and Tabuada and Pappas (2006) for discrete-time control systems, in Tabuada (2008) for continuous-time control systems and in Lafferriere, Pappas, and Sastry (2000) for o-minimal hybrid systems among others. Reduction of continuous control systems to continuous control systems with lower dimensional state space has been addressed in Grasse (2007), Pola, van der Schaft, and Di Benedetto (2006), Tabuada and Pappas (2004) and van der Schaft (2004);
(ii) *Quantized control systems*: finite abstractions have been studied in Bicchi, Marigo, and Piccoli (2002, 2006) for certain classes of control systems with quantized inputs;

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(iii) *Qualitative reasoning*: symbolic models were constructed using methods of qualitative reasoning in Kuipers (1994) and Ramamoorthy and Kuipers (2003);
(iv) *Stochastic automata*: abstractions of continuous-time control systems by means of stochastic automata have been studied in Lunze and Nixdorf (2001) and Schroder (2003).

We defer to the last section of the paper a comparison between the results presented in this paper and the above lines of research. In this paper we follow the line of research based on simulation/bisimulation by making use of the recently introduced notion of *approximate bisimulation* (Girard & Pappas, 2007), that captures equivalence of systems in an approximate setting. By relaxing the usual notion of bisimulation to approximate bisimulation, a larger class of control systems can be expected to admit symbolic models. In fact the work in Tabuada (2008) shows that for every asymptotically stabilizable control system it is possible to construct a symbolic model, which is based on an approximate notion of simulation (one-sided version of bisimulation). However, if a controller fails to exist for the symbolic model, nothing can be concluded regarding the existence of a controller for the original model. This drawback is a direct consequence of the one-sided notion used in Tabuada (2008). For this reason, an extension of the results in Tabuada (2008) from simulation to bisimulation is needed. The aim of this paper is precisely to provide such extension. The key idea in the results that we propose is to replace the assumption of asymptotic stabilizability of Tabuada (2008) with the stronger notion of asymptotic stability. *We show that every incrementally globally asymptotically stable nonlinear control system admits a symbolic model that is an approximate bisimulation, with a precision that is a-priori defined, as a design parameter. Furthermore, if the state space of the control system is bounded, the symbolic model is finite. Moreover, for incrementally input-to-state stable digital control systems, i.e. systems where control signals are piecewise-constant, a symbolic model can be obtained by quantizing the space of inputs.* As an illustrative example, we apply the proposed techniques to a control design problem for a pendulum. A preliminary version of these results appeared in Pola, Girard, and Tabuada (2007).

2. Control systems and stability notions

2.1. Notations

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+_0$ denote the natural, integers, real, positive and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$ we denote by x' the transpose of x and by x_i the i-th element of x; furthermore $\|x\|$ denotes the infinity norm of x; we recall that $\|x\| := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, where $|x_i|$ is the absolute value of x_i. The symbol $\mathcal{B}_\varepsilon(x)$ denotes the closed ball centered at $x \in \mathbb{R}^n$ with radius $\varepsilon \in \mathbb{R}^+_0$, i.e. $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\}$. For any $A \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}^+$ define $[A]_\mu := \{a \in A \mid a_i = k_i\mu, k_i \in \mathbb{Z}, i = 1, \ldots, n\}$. The set $[A]_\mu$ will be used in the subsequent developments as an approximation of the set A with precision μ. By geometrical considerations on the infinity norm, for any $\mu \in \mathbb{R}^+$ and $\lambda \geq \mu/2$ the collection of sets $\{\mathcal{B}_\lambda(q)\}_{q\in[\mathbb{R}^n]_\mu}$ is a covering of $\mathbb{R}^n$, i.e. $\mathbb{R}^n \subseteq \bigcup_{q\in[\mathbb{R}^n]_\mu} \mathcal{B}_\lambda(q)$; conversely for any $\lambda < \mu/2$, $\mathbb{R}^n \nsubseteq \bigcup_{q\in[\mathbb{R}^n]_\mu} \mathcal{B}_\lambda(q)$.

We now recall from Khalil (1996) and Sontag (1998) some notions that will be employed in Sections 2.2 and 2.3 to define trajectories and some stability notions for control systems. A function $f : [a, b] \to \mathbb{R}^n$ is said to be absolutely continuous on $[a, b]$ if for any $\varepsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ so that for every $k \in \mathbb{N}$ and for every sequence of points $a \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq b$, if $\sum_{i=1}^m (b_i - a_i) < \delta$ then $\sum_{i=1}^m |f(b_i) - f(a_i)| < \varepsilon$. A function $f :]a, b[\to \mathbb{R}^n$ is said to be locally absolutely continuous if the restriction of f to any compact subset of $]a, b[$ is absolutely continuous. Given a measurable function $f : \mathbb{R}^+_0 \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$; f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. A continuous function $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. The following notions will be used in Sections 3–5 to define the concept of approximate bisimulation and the symbolic models that we propose in this paper. The identity map on a set A is denoted by 1_A. Given two sets A and B, if A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \to B$ the symbol $f(A)$ denotes the image of A through f, i.e. $f(A) := \{b \in B : \exists a \in A \text{ s.t. } b = f(a)\}$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ if and only if $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation of R, i.e. $R^{-1} := \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2. Control systems

The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1. A *control system* is a quadruple $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $U \subseteq \mathbb{R}^m$ is the input space;
- $\mathcal{U}$ is a subset of the set of all locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $K \subset \mathbb{R}^n$, there exists a constant $\kappa > 0$ such that $\|f(x, u) - f(y, u)\| \leq \kappa\|x - y\|$, for all $x, y \in K$ and all $u \in U$.

A locally absolutely continuous curve $\mathbf{x} :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\mathbf{u} \in \mathcal{U}$ satisfying $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$, for almost all $t \in]a, b[$. Although we have defined trajectories over open domains, we shall refer to trajectories $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\mathbf{z} :]a, b[\to \mathbb{R}^n$ such that $\mathbf{x} = \mathbf{z}|_{[0,\tau]}$. We will also write $\mathbf{x}(t, x, \mathbf{u})$ to denote the point reached at time $t \in]a, b[$ under the input $\mathbf{u}$ from initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories (Sontag, 1998).

A control system Σ is said to be *forward complete* if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in Angeli and Sontag (1999). Simpler, but only sufficient, conditions for forward completeness are also available in the literature. These include linear growth or compact support of the vector field (see e.g. Lee and Markus (1967)).

2.3. Stability notions

The results presented in this paper will assume certain stability assumptions that we briefly recall in this section.

Definition 2.2 (*Angeli, 2002*). A control system Σ is *incrementally globally asymptotically stable* (δ-GAS) if it is forward complete and there exists a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}^+_0$, any $x, y \in \mathbb{R}^n$ and any $\mathbf{u} \in \mathcal{U}$ the following condition is satisfied:

$$\|\mathbf{x}(t, x, \mathbf{u}) - \mathbf{x}(t, y, \mathbf{u})\| \leq \beta(\|x - y\|, t). \tag{1}$$