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Brief paper

Capability and limitation of max- and sum-type construction of Lyapunov functions for networks of iISS systems[☆]

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ABSTRACT

This paper addresses the problem of verifying stability of networks whose subsystems admit dissipation inequalities of integral input-to-state stability (iISS). We focus on two ways of constructing a Lyapunov function satisfying a dissipation inequality of a given network. Their difference from one another is elucidated from the viewpoint of formulation, relation, fundamental limitation and capability. One is referred to as the max-type construction resulting in a Lipschitz continuous Lyapunov function. The other is the sum-type construction resulting in a continuously differentiable Lyapunov function. This paper presents geometrical conditions under which the Lyapunov construction is possible for a network comprising $n \geq 2$ subsystems. Although the sum-type construction for general $n > 2$ has not yet been reduced to a readily computable condition, we obtain a simple condition of iISS small gain in the case of $n = 2$. It is demonstrated that the max-type construction fails to offer a Lyapunov function if the network contains subsystems which are not input-to-state stable (ISS).

1. Introduction

In order to verify stability of an interconnected system, the notion of input-to-state stability (ISS) is useful for dealing with the subsystems which do not admit a finite linear gain (Sontag, 1989). For example, the ISS small-gain theorem is available for establishing the ISS property of interconnection of two ISS subsystems (Jiang, Teel, & Praly, 1994; Teel, 1996). The notion of integral input-to-state stability (iISS) has been also developed to characterize nonlinear systems which are not finite in the sense of ISS (Angeli, Sontag, & Wang, 2000). For the interconnection of two subsystems, the philosophy of the ISS small-gain theorem has been extended to the iISS case (Ito, 2006; Ito & Jiang, 2009). On the other hand, many practical systems such as logistic systems, biological systems, communication networks and power networks consist of more than two subsystems and have complex interconnection structures. To address such large-scale systems of ever-increasing importance, the ISS small-gain theorem has been extended to the case of general networks recently (Dashkovskiy, Rüffer, & Wirth, 2007; Jiang & Wang, 2008).

The ISS small-gain theorem was originally given in terms of bounds for trajectories. Having Lyapunov functions is sometimes advantageous in the analysis and design of nonlinear systems. A Lyapunov formulation of the ISS small-gain theorem was given in Jiang, Mareels, and Wang (1996) for the first time, and extended to the general networks in Dashkovskiy, Rüffer, and Wirth (2006, 2010) and Liu, Hill, and Jiang (2009). The ISS Lyapunov functions constructed there are defined as the maximum among ISS Lyapunov functions of the subsystems, which directly yield Lipschitz continuous Lyapunov functions of the networks.[2] In contrast, the iISS small-gain theorem developed in Ito (2006), Ito and Jiang (2009) is proved by using the sum of iISS Lyapunov functions of two subsystems, which directly results in continuously differentiable Lyapunov functions. For general networks of ISS subsystems, a problem of finding sum-type Lyapunov functions is formulated in Dashkovskiy, Ito, and

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2 Historically, the max-type and the sum-type construction incorporates the idea of vector and scalar Lyapunov functions, respectively (Michel, 1983; Sandell, Varaiya, Athans, & Safonov, 1978).

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Wirth (2011) although it is solved only in the linear case. In Dashkovskiy, Ito et al. (2011), a max-type Lyapunov function yielding a *dissipative inequality* of the ISS network has been derived from the ISS subsystems defined in the *dissipative form* under a technical assumption. Note that the max-type construction was originally derived in the so-called *implication form* Dashkovskiy et al. (2006, 2010), Jiang et al. (1996), and Liu et al. (2009). The dissipation form has the advantage that it unifies the definition of ISS and iISS systems, while the implication form is invalid for iISS systems which are not ISS. An attempt to tackle iISS networks was made in Rüffer, Kellett, and Weller (2010) which verifies that a new scheme is required for establishing the stability of networks involving non-ISS subsystems.

The purpose of this paper is to deal with subsystems described by dissipative inequalities covering the iISS property, and to elucidate capabilities, limitations and relations of the two constructions of Lyapunov functions for general networks. This paper shows that the max-type construction yields a dissipation inequality of the general network consisting of general n subsystems if a matrix-like small-gain condition holds without any assumption on the interaction with external disturbance. From the sum-type construction, this paper also derives a sufficient condition for the stability of the network. Although the condition has not yet been expressed in a computationally convenient form for general n, it is reduced to a small-gain condition in the case of two subsystems. Moreover, this paper proves that the max-type construction can only deal with ISS subsystems, while the sum-type construction can handle non-ISS as well as ISS subsystems. This paper gives geometrical insights into the capabilities and limitations of the two constructions. In order to avoid confusion, it is made clear here that the focus of this paper is on how to *compose a Lyapunov function for the entire network*, which is independent of another interesting issue of how to formulate interaction between individual subsystems such as sum and maximum (Dashkovskiy, Kosmykov, & Wirth, 2011; Dashkovskiy et al., 2010).[3]

We use the following notation. The symbol $|\cdot|$ stands for the Euclidean norm. A continuous function $\omega : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is said to be positive definite and denoted by $\omega \in \mathcal{P}$ if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. The symbol **Id** denotes the identity map. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Negation is $\neg$. For $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we use the simple notation $\lim f(s) = \lim g(s)$ to describe $\{\lim f(s) = \infty \wedge \lim g(s) = \infty\} \vee \{\infty > \lim f(s) = \lim g(s)\}$. Note that the ∞ case is included. In a similar manner, $\lim f(s) \geq \lim g(s)$ denotes $\{(\lim f(s) = \infty) \vee (\infty > \lim f(s) \geq \lim g(s))\}$. For vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The relations $>, \leq, <$ for vectors are defined in the same manner. The negation of $a \geq b$ is denoted by $a \not\geq b$ and this means that there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. For a function of time t, a dot over its symbol stands for d/dt. A preliminary version of the material in this paper was presented at the 48th IEEE Conference on Decision and Control, December, 2009.

2. Problem statement

Consider a network Σ whose state vector $x(t) = [x_1(t)^T, x_2(t)^T, \ldots, x_n(t)^T]^T \in \mathbb{R}^N$ for $t \in \mathbb{R}_+$ is governed by $\dot{x} = f(x, r)$ and admits the existence of a positive definite and radially unbounded $\mathbb{R}_+$-valued $\mathcal{C}^1$ function $V_i(x_i)$ such that the time derivative of $V_i(x_i(t))$ satisfies

$$\dot{V}_i \leq -\alpha_i(V_i(x_i)) + \sum_{j \neq i} \gamma_{ij}(V_j(x_j)) + \kappa_i(|r|) \tag{1}$$

along the trajectories $x_i(t) \in \mathbb{R}^{N_i}$ for each $i = 1, 2, \ldots, n$. The vector $r(t) \in \mathbb{R}^M$ denotes an exogenous signal. The property (1) is usually called a dissipation inequality of a subsystem Σ_i. It is assumed that $\alpha_i \in \mathcal{K}$, $\gamma_{ij} \in \mathcal{K} \cup \{0\}$ and $\kappa_i \in \mathcal{K} \cup \{0\}$ hold. This assumption means that each subsystem Σ_i defined with the state x_i and the inputs $x_j, j \neq i$, and r is integral input-to-state stable (iISS), and that V_i is an iISS Lyapunov function for the individual subsystem Σ_i for each $i = 1, 2, \ldots, n$. We borrow the notions of ISS and iISS properties from Angeli et al. (2000), Sontag (1989) and Sontag and Wang (1995). Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is guaranteed to be input-to-state stable (ISS), and the function V_i is entitled to be a (dissipative) ISS Lyapunov function. The original definition of iISS and ISS is given in terms of trajectories, which is equivalent to the existence of $\mathcal{C}^1$ iISS and ISS Lyapunov functions, respectively (Angeli et al., 2000; Sontag & Wang, 1995). By definition, an ISS system is always iISS. The converse does not hold.

The objective of this paper is to derive conditions under which the network Σ in total is iISS with respect to input r and state x through construction of an iISS Lyapunov function for the overall network. We want to cover ISS as a special case. To this end, we define operators $A, \Gamma: s \in \mathbb{R}_+^n \mapsto z \in \mathbb{R}_+^n$ by

$$z = A(s) = [\alpha_1(s_1), \alpha_2(s_2), \ldots, \alpha_n(s_n)]^T,$$

$$z = \Gamma(s) = \left[\sum_{j \neq 1} \gamma_{1j}(s_j), \sum_{j \neq 2} \gamma_{2j}(s_j), \ldots, \sum_{j \neq n} \gamma_{n,j}(s_j)\right]^T.$$

The operator $K: \tau \in \mathbb{R}_+ \mapsto z \in \mathbb{R}_+^n$ is defined by

$$z = K(\tau) = [\kappa_1(\tau), \kappa_2(\tau), \ldots, \kappa_n(\tau)]^T.$$

The following vectors are also defined:

$$V(x) = [V_1(x_1), V_2(x_2), \ldots, V_n(x_n)]^T,$$
$$\dot{V} = [\dot{V}_1, \dot{V}_2, \ldots, \dot{V}_n]^T,$$

where $\dot{V}_i = dV_i/dt$ is the time derivative along the trajectories $x_i(t) \in \mathbb{R}^{N_i}$. Then the dissipation inequalities (1) can be compactly written as

$$\dot{V} \leq (-A + \Gamma) \circ V(x) + K(|r|). \tag{2}$$

Recall that the relation $\leq$ for vectors used in (2) is interpreted componentwise. The goal of this paper is to find a positive definite and radially unbounded function $V_{cl} : \mathbb{R}^N \to \mathbb{R}_+$ satisfying the dissipation inequality

$$\dot{V}_{cl} \leq -\alpha_{cl}(V_{cl}(x)) + \kappa_{cl}(|r|) \tag{3}$$

along the trajectories $x(t)$ of the network Σ for some $\alpha_{cl} \in \mathcal{P}$ and $\kappa_{cl} \in \mathcal{K} \cup \{0\}$. The property (3) guarantees that the network Σ is iISS with respect to input r and state x. Furthermore, the network Σ is ISS if $\alpha_{cl} \in \mathcal{K}_\infty$.

Remark 1. The function V_i satisfying (1) is an iISS Lyapunov function even when $\alpha_i \in \mathcal{P}$ (Angeli et al., 2000). Nevertheless, to allow for feedback loops in the network Σ, this paper assumes $\alpha_i \in \mathcal{K}$ which is a strict subset of $\mathcal{P}$. It is proved in Ito (2010a), Ito and Jiang (2009) that a feedback interconnection of iISS systems defined by the dissipation inequalities (1) with $\gamma_{ij} \in \mathcal{K}$ is guaranteed to be iISS only if for each i the function α_i can be bounded from below by a class $\mathcal{K}$ function. The dynamics of interconnected comparison systems also leads us to the same observation for "bounding" systems (Rüffer et al., 2010).

[3] For example, one may want to use maximization instead of the sum on the right side of (1).