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Brief paper

Optimal memoryless control in Gaussian noise: A simple counterexample☆

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ABSTRACT

In this paper, we investigate control strategies for a scalar, one-step delay system in discrete-time, i.e., the state of the system is the input delayed by one time unit. In contrast with classical approaches, here the control action must be a memoryless function of the output of the plant, which comprises the current state corrupted by measurement noise. We adopt a first order state-space representation for the delay system, where the initial state is a Gaussian random variable. In addition, we assume that the measurement noise is drawn from a white and Gaussian process with zero mean and constant variance. Performance evaluation is carried out via a finite-time quadratic cost that combines the second moment of the control signal, and the second moment of the difference between the initial state and the state at the final time. We show that if the time-horizon is one or two then the optimal control is a linear function of the plant's output, while for a sufficiently large horizon a control taking on only two values will outperform the optimal affine solution. This paper complements the well-known counterexample by Hans Witsenhausen, which showed that the solution to a linear, quadratic and Gaussian optimal control paradigm might be nonlinear. Witsenhausen's counterexample considered an optimization horizon with two time-steps (two stage control). In contrast with Witsenhausen's work, the solution to our counterexample is linear for one and two stages but it becomes nonlinear as the number of stages is increased. The fact that our paradigm leads to nonlinear solutions, in the multi-stage case, could not be predicted from prior results. In contrast to prior work, the validity of our counterexample is based on analytical proof methods. Our proof technique rests on a simple nonlinear strategy that is useful in its own right, since it outperforms any affine solution.

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1. Introduction

Consider the following discrete-time system:

$$X(k+1) = U(k), \qquad k \geq 0 \tag{1}$$

$$Y(k) = X(k) + V(k), \qquad k \geq 0 \tag{2}$$

where $V(k)$, $U(k)$, $X(k)$, and $Y(k)$ take values on the reals, and they represent the measurement noise, input, state, and output of the plant, respectively. In addition, we assume that the initial state $X(0)$ is a Gaussian random variable, with zero mean and variance σ_0^2. The measurement noise $\{V(k)\}_{k=0}^{\infty}$ is white, Gaussian, zero mean and with constant variance given by σ_V^2. We also assume that the noise $\{V(k)\}_{k=0}^{\infty}$ and $X(0)$ are mutually independent. In this paper, we will investigate the following problem.

Problem 1. Let σ_0^2 and σ_V^2 be pre-selected positive constants representing the variance of $X(0)$ and $V(k)$, for all $k \in \{0, \ldots, m-1\}$ and m be a given integer denoting the length of an optimization horizon. Consider that the system described by (1)–(2) accepts a control strategy of the following form:

$$U(k) = \mathcal{F}_k(Y(k)), \quad k \in \{0, \ldots, m-1\} \tag{3}$$

where, for each k in the set $\{0, \ldots, m-1\}$, $\mathcal{F}_k : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Given a positive real parameter ϱ, we wish to determine Lebesgue measurable functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that minimize the following cost:

$$\mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \varrho, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} E[(X(m) - X(0))^2] + \varrho \sum_{k=0}^{m-2} E[U(k)^2]. \tag{4}$$

In Fig. 1, we present a graphic interpretation of Problem 1. Notice that Problem 1 can be viewed as an optimal control problem aimed at the design of a memory element capable of storing $X(0)$. The memory element must be constructed using a one-step delay and

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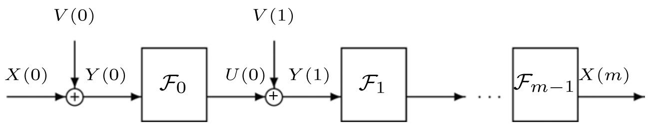

Fig. 1. Graphical interpretation of Problem 1.

memoryless components $\{\mathcal{F}_k\}_{k=0}^{m-1}$, which are used in a feedback configuration. In addition, the memoryless control has access to noisy measurements of the delay's state. Minimizing the cost function defined in (4) amounts to finding the minimal energy memoryless control that leads to the optimal recovery of $X(0)$ from $Y(m-1)$, in a mean square sense.

Paper organization and overview of main results:

The following is the organization of this paper (introduction not included):

- In Section 2, we derive the optimal solution to Problem 1, subject to the constraint that the feedback maps $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. We also show that if m is one or two then affine solutions are optimal.
- In Section 3, we adopt a class of functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that take on only two values for each step k. Given σ_0^2 and σ_V^2, we show that there exists m for which the two valued strategy outperforms the optimal affine control and we provide numerical examples.
- In Section 4, we discuss conclusions and open issues.

1.1. Comparison with related work

The paradigm described in Problem 1 is a linear quadratic and Gaussian optimal control problem. We show that, for up to two stages ($m \in \{1, 2\}$), an optimal solution is attained via affine memoryless control. However, affine solutions are not optimal for all m. In fact, for m large enough, we show that a memoryless control strategy taking on only two values may outperform the optimal affine memoryless controller. The fact that a memoryless policy taking on only two values outperforms the best affine control shows that, for m sufficiently large, the optimal solution to Problem 1 is nonlinear. In fact, for m larger than two, we do not know the optimal solution to Problem 1. This is not surprising, since the related two stage problem suggested by Witsenhausen (1968) remains open forty years after its publication. In Bansal and Basar (1987) and Basar (2008), the authors place the Witsenhausen counterexample within a broad class of dynamic decision problems with nonclassical information. In Grover and Sahai (2008), a vector version of the Witsenhausen counterexample is presented. Moreover, it was reported in Papadimitriou and Tsitsiklis (1986) that the discretized version of Witsenhausen's counterexample is NP-complete. This fact has motivated the numerical studies in Baglietto, Parisini, and Zoppoli (2001), Deng and Ho (1999) and Lee, Lau, and Ho (2001).

The work in Ho and Chu (1972) and Chu (1972), considered the case where a linear information pattern is defined by a directed graph. Using the notion of partially nested information structure, the authors of Chu (1972) and Ho and Chu (1972) characterize when the optimal solution can be found, while bounds are derived when the optimal is unknown. In Rotkowitz (2006), it is shown that if Witsenhausen counterexample is modified using an induced norm then the optimal control is linear. In Yuksel and Tatikonda (2009), the authors show that linear sensing policies over Gaussian channels might not be optimal in a distributed multi-sensor, single controller scenario, for the minimization of a quadratic cost function. This is in contrast with the corresponding single-sensor problem, which does admit an optimal linear solution. The work in Rotkowitz (2008) addresses one follow-up question listed in the paper by Witsenhausen, more specifically, Rotkowitz (2008) discusses the connections between partially nested structures, for which linear controllers are known to be optimal, and quadratically invariant structures, for which the optimal linear control is known to be convex.

2. Optimal affine memoryless control

In this section, we solve Problem 1 under the constraint that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. In particular, we adopt the following steps:

- We start this section by defining an auxiliary problem (Problem 2), in which we adopt the cost $E[(X(0)-X(m))^2]$ subject to an upper bound constraint on $\sum_{k=0}^{m-2} E[U(k)^2]$, where $X(k)$ and $U(k)$ are those defined in Problem 1;
- Proposition 3 solves Problem 2, for two stages, for the special case where the initial noise is set to zero ($V(0) = 0$);
- In Lemma 6, we find the optimal solution to Problem 2, subject to affine memoryless control strategies;
- In Proposition 7, we give the optimal solution of Problem 2, for two stages ($m = 2$), and we show that the optimal memoryless policy is affine;
- The main result of the section is given in Theorem 9, in which the optimal cost of Problem 1 is computed under the constraint that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine for arbitrary m.

Problem 2. Let σ_0^2 and σ_V^2 be pre-selected positive constants representing the variance of $X(0)$ and $V(k)$, for all $k \in \{0, \ldots, m-1\}$ and m be a given integer denoting the length of an optimization horizon. Consider that the system described by (1)–(2) accepts a control strategy of the following form:

$$U(k) = \mathcal{F}_k(Y(k)), \quad k \in \{0, \ldots, m-1\} \tag{5}$$

where, for each k in the set $\{0, \ldots, m-1\}$, $\mathcal{F}_k : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Given a positive real parameter γ, we wish to determine Lebesgue measurable functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that minimize the following cost:

$$\mathcal{C}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} E[(X(m)-X(0))^2] \tag{6}$$

$$\text{s.t.} \sum_{k=0}^{m-2} E[U(k)^2] \le (m-1)\sigma_V^2\gamma. \tag{7}$$

Using standard Lagrangian relaxation (Boyd & Vandenberghe, 2004), there exists a positive real number ϱ, such that the optimal solution of Problem 2, is also an optimal solution of the following problem:

$$\min_{\{\mathcal{F}_k\}_{k=0}^{m-1}} E[(X(m)-X(0))^2] + \varrho \sum_{k=0}^{m-2} E[U(k)^2]$$

with $X(0)$, $X(m)$ and $U(k)$ defined as in Problem 2, where ϱ is the Lagrange multiplier associated with the constraint $\sum_{k=0}^{m-2} E[U(k)^2] \le (m-1)\sigma_V^2\gamma$. Hence, using Lagrangian relaxation we can recover Problem 1. We will show later in Theorem 9, that, subject to affine memoryless control and under some additional conditions, Problems 1 and 2 share an optimal solution. We introduce Problem 2 because it will aid in the solution of Problem 1, subject to affine memoryless control.

The following proposition is an important supporting result for this section. It provides a solution to Problem 2, for the particular case, where m is two and the initial noise is set to zero ($V(0) = 0$). Our proof uses a result in Bansal and Basar (1987), where a similar problem was analyzed. In Fig. 2, we present an alternative interpretation of Proposition 3.

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