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Sufficient conditions for the convergence of a class of nonlinear distributed consensus algorithms☆

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ABSTRACT

This paper is concerned with the convergence of a class of continuous-time nonlinear consensus algorithms for single integrator agents. In the consensus algorithms studied here, the control input of each agent is assumed to be a state-dependent combination of the relative positions of its neighbors in the information flow graph. Using a novel approach based on the smallest order of the nonzero derivative, it is shown that under some mild conditions the convex hull of the agents has a contracting property. A set-valued LaSalle-like approach is subsequently employed to show the convergence of the agents to a common point. The results are shown to be more general than the ones reported in the literature in some cases. An illustrative example demonstrates how the proposed convergence conditions can be verified.

1. Introduction

Consensus is one of the most important specifications in multi-agent control applications. Early work on the consensus problem can be traced back to the field of computer science and distributed computations (Olfati-Saber, Alex Fax, & Murray, 2007). In the classical consensus problem, it is desired to find a state update rule for the agents such that some quantity of interest in every agent converges to a common value in the steady state. Further results on this subject are presented in the literature in the past few years; e.g., see Olfati-Saber and Murray (2004) and Ren and Beard (2005). In Olfati-Saber and Murray (2004), linear time-invariant (LTI) consensus protocols are proposed for multi-agent systems subject to switching communication topologies and time delay. The work Ren and Beard (2005) proposes both discrete- and continuous-time consensus protocols for a group of agents which exchange information over limited and unreliable communication links with time-varying topology. Recently, some algorithms have been proposed in the literature which guarantee the connectivity of the underlying network of agents (Ajorlou, Momeni, & Aghdam, 2010; Dimarogonas & Kyriakopoulos, 2008; Ji & Egerstedt, 2007). Collision avoidance is another important problem concerning the consensus algorithms, and has been addressed in a number of papers (Dimarogonas & Kyriakopoulos, 2008; Olfati-Saber, 2006; Tanner, Jadbabaie, & Pappas, 2007).

In all of the above-mentioned algorithms, the stability of the system under some control strategy is to be determined, typically by finding an appropriate Lyapunov function (Dimarogonas & Kyriakopoulos, 2008; Ji & Egerstedt, 2007; Olfati-Saber, 2006; Tanner et al., 2007). However, constructing a proper Lyapunov function is known to be cumbersome, in general. Motivated by this shortcoming, some recent papers have considered the stability of general distributed consensus algorithms (Lin, Francis, & Maggiore, 2005, 2007; Moreau, 2004, 2005). Graphical conditions are presented in Moreau (2004) for the exponential stability of a class of continuous linear time-varying (LTV) systems whose state-space matrix is Metzler with zero row sums. In Moreau (2005), the convergence of discrete-time nonlinear consensus algorithms with time-dependent communication links is shown under a convexity assumption and some conditions on the communication graph. As the continuous-time counterpart of Moreau (2005), the work Lin et al. (2007) studies the state agreement for coupled nonlinear differential equations with switching vector fields and topology. It is shown that under a strict sub-tangentiality condition and uniformly quasi-strongly connectivity of the interaction digraph, the system has the property of asymptotic state agreement. Somewhat relaxed conditions for the case of a static interaction digraph are presented in Lin et al. (2005). Nonlinear consensus algorithms arise in applications where other design criteria such as connectivity preservation and collision avoidance are to be satisfied during the convergence to consensus

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(Ajorlou et al., 2010; Dimarogonas & Kyriakopoulos, 2008; Ji & Egerstedt, 2007).

The present paper studies the convergence of a class of continuous-time nonlinear consensus algorithms for single integrator agents. The information flow graph of the agents is assumed to be static and directed. The control input of each agent is considered as a state-dependent combination of the relative positions of its neighbors in the information flow graph. Sufficient conditions are provided which guarantee the convergence of the agents to a common point for this class of consensus algorithms. It is shown that under some mild conditions, the convex hull of the agents has a contracting property. This property is used later to prove the convergence of the agents to a common point. The proposed convergence conditions are more general than the ones reported in Lin et al. (2005, 2007) under the additional assumption that the weights are analytic for a static interaction graph. The verification of the proposed convergence conditions is illustrated via a leader-based consensus example.

2. Problem formulation

Definition 1. For a smooth function $f : \mathbb{R} \to \mathbb{R}$, the *index* of f at time t, denoted by $\rho(f(t))$, is defined as the smallest natural number n for which $f^{(n)}(t) \neq 0$. Similarly, the *extended index* of f at time t, denoted by $\tilde{\rho}(f(t))$, is defined as the smallest nonnegative integer n for which $f^{(n)}(t) \neq 0$, where $f^{(0)}(t)$ is defined to be $f(t)$.

Definition 2. A set-valued function $S(\cdot)$ is said to be *nested* if for every $t_1, t_2 \in \mathbb{R}$, where $0 \leq t_1 \leq t_2$, the relation $S(t_2) \subseteq S(t_1)$ holds.

Definition 3. A digraph G is said to be *quasi-strongly connected* if for every two distinct vertices u and v of G, there is a vertex from which both u and v are reachable (see Gondran & Minoux, 1984).

Definition 4. A group of agents $1, \ldots, n$ is said to converge to consensus if $q_i(t) \to \bar{q}$ as $t \to \infty$ for any $i \in \mathbb{N}_n := \{1, \ldots, n\}$, where $q_i(t) \in \mathbb{R}^m$ denotes the state of agent i at time t, and $\bar{q}$ is a constant.

Definition 5. A family $\mathscr{A} = \{A_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to have the finite intersection property if every finite sub-family $\{A_1, A_2, \ldots, A_n\}$ of $\mathscr{A}$ satisfies $\bigcap_{i=1}^{n} A_i \neq \emptyset$ (see Kelley, 1975).

Consider a set of n agents in the 2D plane with single integrator dynamics, i.e.

$$\dot{q}_i(t) = u_i(t), \quad i \in \mathbb{N}_n \tag{1}$$

where $q_i(t) \in \mathbb{R}^2$ represents the position of agent i at time t, and u_i is the corresponding control signal. Note that for brevity, the time argument is omitted hereafter in all time-dependent functions, wherever it is not necessary. Denote with $G = (V, E)$ the information flow graph, with $V = \{1, \ldots, n\}$ representing the set of n vertices (associated with the n agents), and $E \subseteq V \times V$ representing the corresponding edges. The information flow graph G is assumed to be static and directed. There is a directed edge from vertex j to vertex i in G if and only if $(j, i) \in E$. The set of neighbors of vertex i in G is defined as $N_i = \{j | (j, i) \in E\}$, and its indegree is denoted by $d_i = |N_i|$. Each agent is only allowed to incorporate its own position and the position of its neighbors in its control law. In this paper, the distributed control laws of the following form are considered

$$u_i = -\sum_{j \in N_i} \beta_{ij} \times (q_i - q_j), \quad i \in \mathbb{N}_n \tag{2}$$

where the coefficients $\beta_{ij} : \mathbb{R}^{2(d_i+1)} \to \mathbb{R}, i \in \mathbb{N}_n, j \in N_i$, are state-dependent. More specifically, each coefficient β_{ij} is a function of the position of agent i and the positions of the neighbors of agent i in G. The main contribution of this paper is to present sufficient conditions on the coefficients β_{ij} in (2), which guarantee the convergence of the agents to consensus.

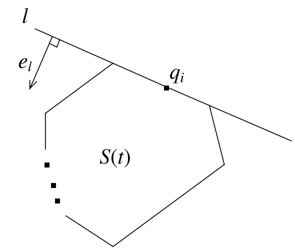

Fig. 1. $S(t)$ is the convex hull of the agents at time t, q_i is the position of an agent on l, and e_l is the unit vector perpendicular to l in the direction of the half-plane containing $S(t)$.

3. Sufficient conditions for convergence

The aim of this section is to show that under the following assumptions on the coefficients β_{ij} in (2), the agents converge to consensus.

Assumption 1. The state-dependent coefficients β_{ij} in (2) are analytic, real and nonnegative for any $i \in \mathbb{N}_n$ and $j \in N_i$.

Assumption 2. The system (1) with the control law of the form (2) has no solution in which the convex hull of the agents is not a singleton and is fixed, with at least one agent being fixed at each vertex.

Lemma 1. *Consider a function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f^{\rho(f(t))}(t) > 0$, for some t. Then, there exists $\delta > 0$ such that $f(t) < f(t + \tau), \forall \tau \in (0, \delta]$. Similarly, if $f^{\rho(f(t))}(t) < 0$, then there exists $\delta > 0$ for which $f(t) > f(t + \tau), \forall \tau \in (0, \delta]$.*

Proof. The proof is straightforward, and is omitted here. □

Denote with $S(t)$ the convex hull of the agents at time t, i.e. $S(t) = \mathrm{Conv}(\{q_i(t) | i \in \mathbb{N}_n\})$. In what follows, a few lemmas are presented in order to prove the nestedness property for $S(t)$. To this end, it is required to investigate the behavior of the agents on the boundary of $S(t)$. Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Note that this intersection will be either an edge or a vertex of $S(t)$ (see Fig. 1 for the case when the intersection is an edge). Denote with e_l the unit vector perpendicular to l, in the direction of the half-plane containing $S(t)$. Define $f_l : \mathbb{R}^2 \to \mathbb{R}$ as $f_l(x) =< x, e_l >$, i.e., the projection of x on e_l. Let agent i be on l at time t. Denote with $N_i^l(t)$ the set of those neighbors of i lying on l, and with $\bar{N}_i^l(t)$ the set of those neighbors not lying on l. Now, define $\eta_{i1}^l(t)$ and $\eta_{i2}^l(t)$ as follows:

$$\eta_{i1}^l(t) = \begin{cases} \min\limits_{j \in N_i^l(t)} \{\tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j))\}, & N_i^l(t) \neq \emptyset \\ \infty, & N_i^l(t) = \emptyset \end{cases} \tag{3}$$

and

$$\eta_{i2}^l(t) = \begin{cases} \min\limits_{j \in \bar{N}_i^l(t)} \{\tilde{\rho}(\beta_{ij})\}, & \bar{N}_i^l(t) \neq \emptyset \\ \infty, & \bar{N}_i^l(t) = \emptyset \end{cases} \tag{4}$$

where in calculating $\tilde{\rho}(\beta_{ij})$, β_{ij} is regarded as an implicit function of time. It is straightforward to verify that $\eta_{i1}^l(t) \geq 1$ and $\eta_{i2}^l(t) \geq 0$. Define also $\eta_i^l(t) = \min\{\eta_{i1}^l(t), \eta_{i2}^l(t)\}$.

Lemma 2. *Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. Then, the following statements are true:*

(i) *If $\eta_i^l = 0$, then $f_l(\dot{q}_i) > 0$.*

(ii) *If $\eta_i^l \geq 1$, then $f_l(q_i^{(k)}) = 0$, for $k = 1, \ldots, \eta_i^l$.*