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A control-theoretic study on iterative solutions to nonlinear equations for applications in embedded systems[☆]

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ABSTRACT

In this paper, the fixed point iteration and Newton's methods for iteratively solving nonlinear equations are studied in the control theoretical framework. This work is motivated by the ever increasing demands for integrating iterative solutions of nonlinear functions into embedded control systems. The use of the well-established control theoretical methods for our application purpose is inspired by the recent control-theoretical study on numerical analysis. Our study consists of two parts. In the first part, the existing fixed point iteration and Newton's methods are analysed using the stability theory for the sector-bounded Lure's systems. The second part is devoted to the modified iteration methods and the integration of sensor signals into the iterative computations. The major results achieved in our study are, besides some academic examples, applied to the iterative computation of the air path model embedded in the engine control systems.

1. Introduction

Iterative computation is one of the standard techniques for solving nonlinear equations (Quarteroni, Sacco, & Saleri, 2000; Stoer & Bulirsch, 2002). It is a powerful mathematical tool widely used in engineering applications (Eich-Soellner & Führer, 1998; Hoffmann, Marx, & Vogt, 2005). Thanks to the rapid development of microprocessor technology, more and more iterative solutions of nonlinear equations are implemented on electrical control units (ECUs) in real-time embedded systems. For instance, for the real-time control and on-board-diagnosis (OBD) of an internal combustion engine numerous iterative computation blocks are integrated into the ECU (Kiencke & Nielsen, 2005).

Nonlinear equations like

$$x = \varphi(x) \quad \text{and} \quad f(x) = 0 \tag{1}$$

are most typical forms met in engineering applications (Eich-Soellner & Führer, 1998; Hoffmann et al., 2005). The so-called fixed point iteration described by

$$x(k+1) = \varphi(x(k))$$

and Newton's methods with the general iterative form

$$x(k+1) = x(k) - \Psi(x(k))f(x(k))$$

are standard algorithms for solving (1) iteratively, where k stands for the iterative number and $\Psi(x(k))$ is some matrix (Quarteroni et al., 2000; Stoer & Bulirsch, 2002). Under certain conditions, the iteration will converge to the solution of the equations x^*, i.e.

$$\lim_{k\to\infty} x(k) = x^*, \quad x^* = \varphi(x^*) \quad \text{or} \quad f(x^*) = 0.$$

For the real-time applications in embedded control systems, the nonlinear equations in (1) build typically sub-models embedded in a complex functional block and thus are often a function of system inputs and parameters (see also the example in Section 5). In this context, the nonlinear equations in (1) can be extended to

$$x = \varphi(x, p) \quad \text{and} \quad f(x, p) = 0 \tag{2}$$

with p being a vector representing these (time-varying) system inputs and parameters. The iterative computation will then be triggered by each update of p. For applications in embedded control systems, such computation often demands for high real-time ability. Although ECUs of the new generation are becoming

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more powerful, the low cost requirement on the one hand and ever increasing demands for the high system performance on the other hand call for more attention to the real-time implementation of iterative algorithms.

Our study presented in this paper is mainly motivated by the real-time implementation of control and OBD algorithms on the ECU for the internal combustion engine control (Weinhold, 2007). In this industrial application, we have been confronted with the following two problems: (a) the convergence rate of the existing iterative solutions for some nonlinear equations does not satisfy the real-time requirement (b) the quantisation errors due to the use of look-up tables instead of analytical functions may considerably affect the computation performance. As a result, poor control performance and, in the worst case, instability can be observed. As investigating solutions for these problems, we notice the recent efforts on applying the modern control theoretical methods to numerical analysis (Bhaya & Kaszkurewicz, 2006, 2007; Kashima & Yamamoto, 2007; Schaerer & Kaszkurewicz, 2001; Söderlind, 2002). In particular, encouraged by the result on the Newton's method reported in Kashima and Yamamoto (2007), we focus our study on (a) the convergence conditions of the fixed point iteration and Newton's methods for nonlinear functions satisfying the so-called sector conditions (Khalil, 2002) (b) the robustness issue with respect to computation errors e.g. caused by the quantisation. In this study, vector-valued nonlinear equations are considered, and the well-established nonlinear control theory (Khalil, 2002) and the LMI (linear matrix inequality) technique (Boyd, Ghaoui, & Feron, 1994) are applied as the main analysis and design tool. This effort allows us to express the convergence conditions in terms of some LMIs, which can then be checked using some standard software.

Driven by the demands for high system performance and reliability, a trend can be observed in the area of embedded control systems that the number of the integrated sensors is continuously increasing. The intuitive idea of our further study on solving nonlinear equations is to integrate those sensor signals, which are available in the embedded system, into the iterative computation. This work has been strongly motivated by the moderate and, in some cases, poor performance delivered by the standard iterative methods as they were applied in the engine control and OBD algorithms running on the ECU. It is reasonable to expect that the additional information provided by the sensors will improve the convergence rate and enhance the robustness. For our purpose, an observer-like iterative method is proposed, which modifies the existing methods and allows us to integrate the available sensor signals so that both the convergence rate and robustness are improved. From the control-theoretical viewpoint, this work is a combination of our study on the existing iterative solutions and the well-established observer design technique.

By testing our new approach to the practical real-time implementation problems, we notice that the sensor noises may have a strong influence on the computation performance. Considering that e.g. in automotive systems the sensors embedded in the control loops and in OBD are often low-cost products, the last topic in our study is dedicated to the robustness analysis with respect to the measurement noises.

The paper is organised as follows. In Section 2, the needed preliminaries in numerical analysis and stability theory are briefly presented, based on which the main problems to be addressed in this paper are formulated. Section 3 is devoted to the control-theoretical study on the existing fixed point iterative solution and Newton's methods with a focus on the convergence and robustness issues. In Section 4, we shall first propose a unified approach for modifying the existing iterative algorithms and integrating sensor signals aiming at improving the convergence rate. It is followed by an analysis of the influence of the measurement noises on the computation performance. The last part in this section deals with the optimisation of the iterative schemes with respect to the convergence rate and robustness. To illustrate the major results, some (academic) examples are included in each section, and an application example from the real-time implementation of the air path model on the ECU is presented in Section 5.

Notation. The notation adopted throughout this paper is fairly standard. $\mathcal{R}^n$ denotes the n-dimensional Euclidean space and $\mathcal{R}^{n\times m}$ the set of all $n \times m$ real matrices. The superscript "T" stands for the transpose of a matrix. "I" and "0" denote the identity and zero matrix with appropriate dimension, respectively. "$P > 0$ (≥ 0)" means matrix P is positive definite (semi-definite). $\bar{\sigma}(\cdot)$, $\underline{\sigma}(\cdot)$ denote the maximum and minimum singular value of a matrix respectively. $\|\cdot\|$ stands for the Euclidean norm. For vector $x \in \mathcal{R}^n$, $\|x\| = \sqrt{x^T x}$. We use $\mathbf{E}(\cdot)$ to denote the expectation of a statistical or stochastic variable.

2. Preliminaries, basic ideas and problem formulation

2.1. Fixed point iteration and Newton's methods

Let the function $\varphi : \mathcal{R}^n \to \mathcal{R}^n$ have a fixed point $x^* : \varphi(x^*) = x^* \in \mathcal{R}^n$. The fixed point iteration algorithm is given e.g. by Stoer and Bulirsch (2002)

$$x(k+1) = \varphi(x(k)) \in \mathcal{R}^n. \tag{3}$$

The following definition and theorem are standard results in numerical analysis (Quarteroni et al., 2000; Stoer & Bulirsch, 2002).

Definition 1. $\varphi : D \subseteq \mathcal{R}^n \to \mathcal{R}^n$ is called contractive on a set $D_o \subset D$ if there exists a constant $\alpha < 1$ such that for all $x, y \in D_o$

$$\|\varphi(x) - \varphi(y)\| \leq \alpha \|x - y\|. \tag{4}$$

Theorem 1 (*Eich-Soellner and Führer (1998), Banach's Fixed Point Theorem*)**.** *Let $\varphi : D \subseteq \mathcal{R}^n \to \mathcal{R}^n$ be contractive in a closed set $D_o \subset D$ and suppose $\varphi(D_o) \subset D_o$. Then φ has a unique fixed point $x^* \in D_o$. Moreover, for any $x(0) = x_0 \in D_o$, the iteration (3) converges to x^*. The distance to the solution is bounded by*

$$\|x^* - x(k)\| \leq \frac{\alpha^k}{1-\alpha} \|x(1) - x(0)\|. \tag{5}$$

It is worth mentioning that under the same conditions given in the above theorem it is easy to prove that

$$\|x^* - x(k)\| \leq \alpha^k \|x^* - x(0)\|. \tag{6}$$

Let the function $f : \mathcal{R}^n \to \mathcal{R}^n$ have a unique solution $x^* \in \mathcal{R}^n$: $f(x^*) = 0$. The standard Newton's method is an iterative algorithm described by

$$x(k+1) = x(k) - (Df(x(k)))^{-1} f(x(k)) \tag{7}$$

with $Df(x(k)) \in \mathcal{R}^{n\times n}$ as the Jacobian matrix of $f(x(k))$. The following theorem describes the major properties of Newton's method (Stoer & Bulirsch, 2002).

Theorem 2. *Given $f : D \subseteq \mathcal{R}^n \to \mathcal{R}^n$ and the convex set $D_o \subseteq D$, let f be differentiable for all $x \in D_o$ and continuous for all $x \in D$. For $x_o \in D_o$ let positive constants r, α, β, γ be given with the following properties:*

$$S_r(x_o) = \{x \,|\, \|x - x_o\| < r\} \subseteq D_o,$$

$$h = \alpha\beta\gamma/2 < 1, \qquad r = \frac{\alpha}{1-h},$$

and let $f(x)$ have the following properties: