Automatica 45 (2009) 1536–1541

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Estimating the domain of attraction for non-polynomial systems via LMI optimizations☆

Graziano Chesi *

*Department of Electrical and Electronic Engineering, University of Hong Kong, Hong Kong*

ARTICLE INFO

*Article history:*
Received 29 May 2008
Received in revised form
9 February 2009
Accepted 17 February 2009
Available online 18 March 2009

*Keywords:*
Nonlinear system
Equilibrium point
Domain of attraction
Lyapunov function
LMI

ABSTRACT

This paper proposes a strategy for estimating the domain of attraction (DA) for non-polynomial systems via Lyapunov functions (LFs). The idea consists of converting the non-polynomial optimization arising for a chosen LF in a polynomial one, which can be solved via LMI optimizations. This is achieved by constructing an uncertain polynomial linearly affected by parameters constrained in a polytope which allows us to take into account the worst-case remainders in truncated Taylor expansions. Moreover, a condition is provided for ensuring asymptotical convergence to the largest estimate achievable with the chosen LF, and another condition is provided for establishing whether such an estimate has been found. The proposed strategy can readily be exploited with variable LFs in order to search for optimal estimates. Lastly, it is worth remarking that no other method is available to estimate the DA for non-polynomial systems via LMIs.


## 1. Introduction

It is well known that estimating the domain of attraction (DA) of an equilibrium point is a problem of fundamental importance in systems engineering. It is also well known that the DA is a complicated set which does not admit an analytic representation in most cases; see for example Genesio, Tartaglia, and Vicino (1985), Vannelli and Vidyasagar (1985) and Chiang, Hirsch, and Wu (1988); Khalil (2001). Therefore, looking for inner approximations of the DA via estimates with simple shape has been a fundamental issue for a long time. A common way of obtaining such estimates is based on Lyapunov stability theory. Specifically, given a Lyapunov function (LF) proving local asymptotical stability of the equilibrium, any sublevel set of this function included in the region where its temporal derivative takes negative values is guaranteed to be an inner estimate of the DA.

In recent years, various and efficient methods have been developed for estimating the DA for polynomial systems, mainly based on LMI relaxations for solving polynomial optimizations (Chesi, 2004a, 2007a; Chesi, Tesi, Vicino, & Genesio, 1999; Jarvis-Wloszek, Feeley, Tan, Sun, & Packard, 2003; Tan & Packard, 2008). These methods are interesting because they amount to solving a convex optimization with LMI constraints for a chosen LF, and because their conservatism can be arbitrarily decreased by increasing the degree of relaxation. See also Chesi and Henrion (in press) and papers therein.

Unfortunately, most real systems are non-polynomial systems; consider, for instance, pendulums, chemical reactors, and systems with saturations. Some methods have been proposed to establish whether an equilibrium point of a non-polynomial system is stable, such as Papachristodoulou and Prajna (2005) which proposes some changes of coordinate, and Mastellone, Hokayem, Abdallah, and Dorato (2004) which proposes the use of polynomial approximations. However, no method has been proposed to estimate the DA for non-polynomial systems via LMIs, which is still an open problem.

This paper proposes a strategy for estimating the DA via LMI optimizations. The idea consists of converting the non-polynomial optimization arising for a chosen LF in a polynomial one, which can be solved through convex optimizations based on LMI relaxations. This polynomial optimization is obtained by expressing the non-polynomial terms via truncated Taylor expansions and parameterizing their remainders inside a convex polytope. This allows us to take into account the worst-case remainders by simply considering only the vertices of the polytope. Moreover, the conservatism of the proposed strategy is investigated, in particular by proposing a condition which ensures asymptotical convergence to the largest estimate achievable with the chosen LF, and another condition which allows one to establish

☆ A preliminary version of this paper was presented in [Chesi, G.(2005). Domain of attraction: Estimates for non-polynomial systems via LMIs. In *16th IFAC world congress on automatic control.*]. The material in this paper was partially presented as a preliminary version at the IFAC World Congress on Automatic Control, Prague, Czech Republic, 2005. This paper was recommended for publication in revised form by Associate Editor Zongli Lin, under the direction of Editor Andrew R. Teel.

* Tel.: +852 22194362; fax: +852 25598738.
*E-mail address:* chesi@eee.hku.hk.

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.02.011

whether such an estimate has been found. The proposed strategy can be readily exploited with variable LFs in order to search for optimal estimates.

This paper improves and extends our preliminary version (Chesi, 2005), in particular by adopting a polytopic parameterization of the remainders of the truncated Taylor expansions and by proposing conditions for non-conservatism. The paper is organized as follows. Section 2 introduces the problem formulation. Section 3 describes the proposed strategy. Section 4 presents some illustrative examples. Lastly, Section 5 reports some final remarks.

## 2. Preliminaries

### 2.1. Problem formulation

Notation

$\mathbb{N}, \mathbb{R}$: sets of natural and real numbers; $0_n$: origin of $\mathbb{R}^n$; $I_n$: identity matrix $n \times n$; $A'$: transpose of matrix $A$; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix $A$; $\mathrm{tr}(A)$: trace of matrix $A$; $\mathrm{ver}(\mathcal{P})$: set of vertices of the polytope $\mathcal{P}$; s.t.: subject to.

Let us consider the continuous-time nonlinear system

$$\begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{r} g_i(x(t))\xi_i(x_{a_i}(t)) \\ x(0) = x_{\text{init}} \end{cases} \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))' \in \mathbb{R}^n$ is the state, $x_{\text{init}} \in \mathbb{R}^n$ is the initial condition, the functions $f, g_1, \ldots, g_r : \mathbb{R}^n \to \mathbb{R}^n$ are polynomial, $a_1, \ldots, a_r \in \{1, \ldots, n\}$ are indexes, and the functions $\xi_i : \mathbb{R} \to \mathbb{R}$ are non-polynomial.

It is assumed that the origin is the equilibrium point of interest. The DA of the origin is the set of initial conditions for which the state asymptotically converges to the origin, and it is indicated by

$$\mathcal{D} = \left\{ x_{\text{init}} \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = 0_n \right\}. \tag{2}$$

In what follows the dependence on the time $t$ will be omitted for ease of notation.

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, positive definite and radially unbounded function, and suppose that $v$ is an LF for the origin in (1), i.e. the time derivative of $v$ along the trajectories of (1) is locally negative definite. Then, the sublevel set

$$\mathcal{V}(c) = \left\{ x \in \mathbb{R}^n : v(x) \leq c \right\} \setminus \{0_n\} \tag{3}$$

is an estimate of $\mathcal{D}$ if

$$\dot{v}(x) < 0 \quad \forall x \in \mathcal{V}(c). \tag{4}$$

Let us assume that $v$ is polynomial of degree $2m$. The problems addressed in this paper are as follows:

(1) for a chosen LF, computing the largest estimate $\mathcal{V}(c^*)$ where

$$c^* = \sup \{ c \in \mathbb{R} : (4) \text{ holds} \}; \tag{5}$$

(2) for a variable LF, searching for optimal estimates according to a chosen criterion, i.e. solving

$$\varrho^*_{2m} = \sup_{v \in \mathcal{F}_{2m}} \zeta(\mathcal{V}(c^*)) \tag{6}$$

where $\mathcal{F}_{2m}$ is the set of polynomials of degree $2m$ (in $n$ variables), and $\zeta : \mathbb{R}^n \to \mathbb{R}$ is a measure of $\mathcal{V}(c^*)$ representing the chosen criterion.

In what follows we will assume that the first $\delta$ derivatives of $\xi_i(x_{a_i})$ are continuous on

$$\mathcal{V}_{a_i}(c) = \left\{ x_{a_i} \in \mathbb{R} : x \in \mathcal{V}(c) \right\}. \tag{7}$$

**Remark 1.** It should be remarked that solving (5) is an unavoidable step for solving (6).

### 2.2. Polynomial optimization via LMIs

Before proceeding let us briefly describe how LMIs can be used for solving polynomial optimizations. Let $x^{\{m\}}$ be a vector containing all monomials of degree less than or equal to $m$ in $x$. Then, $v$ can be written as

$$v(x) = x^{\{m\}'} (V + L(\alpha))\, x^{\{m\}} \tag{8}$$

where $V$ is any symmetric matrix such that $v = x^{\{m\}'} V x^{\{m\}}$, $L$ is any linear parameterization of the set $\mathcal{L} = \{L = L' : x^{\{m\}'} L x^{\{m\}} = 0\}$, and $\alpha$ is a free vector. This representation is known as a Gram matrix (Choi, Lam, & Reznick, 1995) and a square matricial representation (SMR) (Chesi et al., 1999). The expression (8) was introduced in Chesi et al. (1999) in order to investigate the positivity of polynomials via LMIs: indeed, $v$ is a sum of squares (SOS) of polynomials if and only if

$$\exists \alpha : V + L(\alpha) \geq 0 \tag{9}$$

which is an LMI feasibility test (see for instance Chesi (2007b) for the gap between positive polynomials and SOS of polynomials). This can be used in the search for estimates of the domain of attraction in the case of polynomial systems and LFs. Indeed, in such a case the sublevel set $\mathcal{V}(c)$ satisfies (4) if there exists a polynomial $s$ such that

$$s \text{ and } -\dot{v} + (v - c)s \text{ are SOS and vanish only for } x = 0_n \tag{10}$$

which amounts to solving an LMI feasibility test. The degree of $s$ is initially selected small, typically equal to the minimum even value which does not increase the degree of the polynomial $-\dot{v}+(v-c)s$. If this choice does not allow one to compute $c^*$ (such a condition can be detected for instance as done in Chesi (2004b) and Chesi, Garulli, Tesi, and Vicino (2003)), then one increases this degree and repeats the computation.

Analogously, (4) can be investigated via LMIs through moment relaxations (Henrion & Lasserre, 2006) which are dual to SOS relaxations. See also the Matlab toolboxes GloptiPoly and SOSTOOLS where these relaxations are implemented.

## 3. Estimate computation

### 3.1. Fixed Lyapunov function

Let $v$ be an LF satisfying the assumption in Section 2.1. Let $k$, $k \leq \delta$, be a positive integer, and let us rewrite $\xi_i$ via a Taylor expansion of degree $k$ as

$$\xi_i(x_{a_i}) = h_i(x_{a_i}) + w_i \frac{x_{a_i}^k}{k!} \tag{11}$$

where $w_i \in \mathbb{R}$ is a parameter to be selected and

$$h_i(x_{a_i}) = \sum_{j=0}^{k-1} \left. \frac{\mathrm{d}^j \xi_i(x_{a_i})}{\mathrm{d}x_{a_i}^j} \right|_{x_{a_i}=0} \frac{x_{a_i}^j}{j!}. \tag{12}$$

Let us introduce the polynomials

$$p(x) = \frac{\partial v(x)}{\partial x} \left( f(x) + \sum_{i=1}^{r} g_i(x) h_i(x_{a_i}) \right) \tag{13}$$

$$q_i(x) = \frac{\partial v(x)}{\partial x} g_i(x) \frac{x_{a_i}^k}{k!} \tag{14}$$

$$q(x) = (q_1(x), \ldots, q_r(x))'. \tag{15}$$

**Theorem 1.** *Let $c_k$ be the solution of the polynomial optimization*

$$c_k = \sup_{c \in \mathbb{R}} c \quad \text{s.t.} \quad p(x) + q(x)'w < 0 \quad \forall x \in \mathcal{V}(c)\ \forall w \in \mathrm{ver}(\mathcal{W}) \tag{16}$$

*where $\mathcal{W}$ is the rectangle*