Automatica 45 (2009) 1584–1587

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

# A new perspective on criteria and algorithms for reachability of discrete-time switched linear systems[☆]

Zhijian Ji[a,*], Hai Lin[b], Tong Heng Lee[b]

[a] School of Automation Engineering, Qingdao University, Qingdao, 266071, China

[b] Department of Electrical and Computer Engineering, National University of Singapore, 117576, Singapore

ARTICLE INFO

*Article history:*
Received 12 October 2008
Received in revised form
8 January 2009
Accepted 13 February 2009
Available online 25 March 2009

*Keywords:*
Switched systems
Controllability
Reachability
Switching sequence

ABSTRACT

The paper presents a unified perspective on geometric and algebraic criteria for reachability and controllability of controlled switched linear discrete-time systems. Direct connections between geometric and algebraic criteria are established as well as that between the subspace based controllability/reachability algorithm and Kalman-type algebraic rank criteria. Also the existing geometric criteria is simplified and new algebraic conditions on controllability and reachability are given.

© 2009 Elsevier Ltd. All rights reserved.

## 1. Introduction

Switched systems are control systems that consist of a finite number of subsystems and a logical rule that orchestrates switchings among them. The last decade has witnessed a growing interest in the study of such systems because the study is significant from both practical and theoretical point of view (DeCarlo, Branicky, Pettersson, & Lennartson, 2000; Liberzon & Morse, 1999; Sun & Ge, 2005). A challenging topic in switched systems is to evaluate the effect of switched control on the system operation, which is usually formulated as the controllability problem (Krastanov & Veliov, 2005; Petreczky, 2006a; Yang, 2002). The switching mechanism involved in the controllability and reachability was analyzed in Ji, Feng, and Guo (2007), Ji, Wang, and Guo (2007), Ji, Wang, and Guo (2008), Sun (2004), Sun, Ge, and Lee (2002) and Xie and Wang (2003a). Most results along this line were expressed in terms of geometric symbols (e.g. Cheng, Lin, and Wang (2006), Ge, Sun, and Lee (2001), Sun and Ge (2005), Sun et al. (2002), Sun and Zheng (2001), Xie and Wang (2003a)), while a few others algebraic (e.g. Stikkel, Bokor, Szabó (2004) and Yang (2002)). The geometric criteria have the advantage of a straightforward characterization of the reachable/controllable subspace, while the algebraic criteria can be checked and manipulated more conveniently. It is worth noting that there is a lack of systematic perspective on the connections between these two kinds of results as well as the relevant subspace-based algorithms. This motivates the study in this note. Also, the study is fueled by providing computational tools for reachable/controllable subspace of switched linear discrete-time systems. We present not only the aforementioned connections but also some improved geometric and algebraic criteria. Also the relationship between the existing subspace-based algorithms is revealed, which leads to a simplified computation method for controllable subspace. It should be noted that there is a strong relationship between reachability and minimality of linear switched systems (Petreczky, 2006b, 2007). In fact, the presented characterizations of reachability in this note can also be used for devising characterization of minimality of switched linear systems.

The paper is organized as follows: Section 2 presents some preliminary definitions and supporting lemmas. A unified perspective on reachability and controllability criteria is given in Section 3. A brief conclusion is made in Section 4.

## 2. Definitions and supporting lemmas

A switched linear discrete-time system is described by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \qquad (1)$$

[☆] This paper was presented at Proceedings of the 27th Chinese Control Conference, July 16–18, 2008, Kunming, Yunnan, China. This paper was recommended for publication in revised form by Associate Editor Hideaki Ishii under the direction of Editor André L. Tits.

* Corresponding author. Tel.: +86 532 84927689; fax: +86 532 82972727.
*E-mail addresses:* jizhijian@pku.org.cn (Z. Ji), elelh@nus.edu.sg (H. Lin), eleleeth@nus.edu.sg (T.H. Lee).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.02.024

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ the input, $\sigma(k) : \{0, 1, \ldots\} \to \Lambda := \{1, \ldots, m\}$ is the switching path to be designed, and matrix pairs $(A_k, B_k)$ for $k \in \Lambda$ are referred to as the subsystems of (1). Moreover, $\sigma(k) = i$ implies that the $i$th subsystem $(A_i, B_i)$ is activated. Throughout the paper, we assume that the discrete-time switched system (1) is *reversible*, i.e., $A_i$ is nonsingular for all $i \in \Lambda$. The derivation of the following Lemma 2 is based on this assumption (see, e.g. Ge et al. (2001) and Xie and Wang (2003b)).

For any positive integer $k$, set $\underline{k} = \{0, \ldots, k-1\}$. Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, a corresponding switching path $\sigma(k) : \underline{k} \to \Lambda$ is determined by

$$\sigma(0) = \sigma(1) = \cdots = \sigma(h_0 - 1) = i_0$$
$$\sigma(h_0) = \sigma(h_0 + 1) = \cdots = \sigma(h_0 + h_1 - 1) = i_1$$
$$\vdots$$
$$\sigma\left(\sum_{j=0}^{s-2} h_j\right) = \sigma\left(\sum_{j=0}^{s-2} h_j + 1\right) = \cdots = \sigma\left(\sum_{j=0}^{s-1} h_j - 1\right) = i_{s-1}.$$

**Definition 1.** State $x$ is reachable, if there exist a time instant $k > 0$, a switching path $\sigma : \underline{k} \to \Lambda$, and inputs $u : \underline{k} \to \mathbb{R}^p$, such that $x(0) = 0$, and $x(k) = x$. The reachable set of system (1) is the set of states which are reachable. System (1) is said to be (completely) reachable, if its reachable set is $\mathbb{R}^n$.

The controllability counterpart of Definition 1 can be given by replacing '$x(0) = 0$, and $x(k) = x$' with '$x(0) = x$, and $x(k) = 0$'. Given a matrix $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$, we denote $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{n} A^{i-1}\mathscr{W}$. It follows that $\langle A|\mathscr{W}\rangle$ is a minimum $A$-invariant subspace that contains $\mathscr{W}$. Define the subspace sequence $\mathscr{P}_j = \sum_{i=1}^{j} A^{i-1}\mathscr{W}$, $j = 1, 2, \ldots$. Clearly, $\langle A|\mathscr{W}\rangle = \mathscr{P}_n$. Let $\vartheta$ be the integer such that $\vartheta = \min\{j \mid \mathscr{P}_j = \mathscr{P}_{j+1},\ j = 1, 2, \ldots\}$. In association with $A$, we denote by $\rho(A)$ the degree of its minimal polynomial.

**Lemma 1** (*Chen, Desoer, Niederlinski, & Kalman, 1966*). *Given a matrix $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$, $\mathscr{P}_j = \mathscr{P}_\vartheta$ holds for all $j \geq \vartheta$, with $\vartheta$ satisfying $\vartheta \leq \min\{n - \dim \mathscr{W} + 1, \rho(A)\}$.*

An immediate consequence of this lemma is $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{\vartheta} A^{i-1}\mathscr{W}$. For the convenience of statement, we hereafter call $\vartheta$ the *$(A, \mathscr{W})$-invariant subspace index.* For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, set $\mathscr{B} := \operatorname{Im} B$. To study the reachability and controllability of discrete-time switched linear systems, the following recursively defined subspace sequence was introduced in Ge et al. (2001) and Sun et al. (2002).

$$\mathscr{V}_1 = \sum_{s=1}^{m} \mathscr{B}_s, \qquad \mathscr{V}_i = \sum_{s=1}^{m} \langle A_s|\mathscr{V}_{i-1}\rangle, \quad i = 2, 3, \ldots. \tag{2}$$

The subspace $\mathscr{V}$ is defined by $\mathscr{V} = \sum_{i=1}^{\infty} \mathscr{V}_i$. Furthermore, the following elegant result holds.

**Lemma 2** (*Ge et al., 2001; Xie & Wang, 2003b*). *For discrete-time switched linear systems (1), $\mathscr{T} = \mathscr{V} = \mathfrak{C}$, where $\mathscr{T}$ is the set of all reachable states of system (1) and $\mathfrak{C}$ is the set of all controllable states.*

Denote by $\mathbb{d}_i$ the dimension of $\mathscr{V}_i$, i.e., $\mathbb{d}_i = \dim \mathscr{V}_i$. Let $\mu = \min\{i| \mathscr{V}_i = \mathscr{V}_{i+1}, i = 1, 2, \ldots\}$. It can be readily seen that $\mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_\mu$, and $\mathscr{V} = \mathscr{V}_\mu$. Obviously, $\mu$ is fixed once the switched system (1) is given. Furthermore $\mu \leq n - \mathbb{d}_1 + 1$. Hereafter, we call $\mu$ the *joint invariant subspace index of $(A_1, \ldots, A_m;\ B_1, \ldots, B_m)$.*

## 3. A unified perspective on reachability and controllability criteria

### 3.1. Geometric and algebraic criteria

In this subsection we derive at first a simplified geometric criterion for reachability and controllability. Then the corresponding algebraic criterion is given. Finally the geometric and algebraic criteria are discussed from a unified point of view.

Let $\omega_{i,j}$ be the $(A_i, \mathscr{B}_j)$-invariant subspace index, $i, j = 1, \ldots, m$; and $\vartheta_{i,j}$ be the $(A_i, \mathscr{V}_j)$-invariant subspace index, $i = 1, \ldots, m; j = 1, \ldots, \mu - 1$. Set $\vartheta_i = \max\{\omega_{i,s}, \vartheta_{i,j};\ s = 1, \ldots, m; j = 1, \ldots, \mu - 1\}$; and define $\underline{\vartheta_i} \triangleq \{0, 1, \ldots, \vartheta_i - 1\}$, $i = 1, \ldots, m$. We have the following result.

**Theorem 1.** *The switched linear discrete-time system (1) is reachable if and only if $\mathfrak{M} = \mathbb{R}^n$, where*

$$\mathfrak{M} \triangleq \sum_{i_0, \ldots, i_{\mu-1} \in \Lambda}^{j_1 \in \underline{\vartheta_{i_1}}, \ldots, j_{\mu-1} \in \underline{\vartheta_{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathscr{B}_{i_0}. \tag{3}$$

**Proof 1.** Denote by $\rho(A_i)$ the degree of the minimal polynomial of $A_i$. It follows from Lemma 1 that $\mathscr{V}_i$, $i = 2, \ldots, \mu$, can be written in the form

$$\mathscr{V}_i = \sum_{s=1}^{m} \langle A_s|\mathscr{V}_{i-1}\rangle = \sum_{s=1}^{m} \sum_{j=1}^{\vartheta_{s,i-1}} A_s^{j-1}\mathscr{V}_{i-1}, \tag{4}$$

where $\vartheta_{s,i-1}$ is the $(A_s, \mathscr{V}_{i-1})$-invariant subspace index, satisfying

$$\vartheta_{s,i-1} \leq \min\{n - \mathbb{d}_{i-1} + 1, \rho(A_s)\}, \tag{5}$$

with $s = 1, \ldots, m$; $i = 2, \ldots, \mu$; and $\omega_{i,j}$ satisfying

$$\omega_{i,j} \leq \min\{n - \dim \mathscr{B}_j + 1, \rho(A_i)\}. \tag{6}$$

Denote $\beta \triangleq \min\{\dim \mathscr{B}_j, j = 1, \ldots, m\}$, $\rho \triangleq \max\{\rho(A_s), s = 1, \ldots, m\}$. By (5) and (6), and $1 \leq \beta \leq \mathbb{d}_1 < \mathbb{d}_2 < \cdots < \mathbb{d}_{\mu-1} < \mathbb{d}_\mu = \dim \mathscr{V}$, one has $\vartheta_i \leq \min\{n - \beta + 1, \rho\}$, $i = 1, \ldots, m$. We then associate with each subsystem matrix $A_i$ a nonnegative integer set $\underline{\vartheta_i}$. Since $\vartheta_s \geq \vartheta_{s,i-1}$, it follows from Lemma 1 that

$$\sum_{j=1}^{\vartheta_{s,i-1}} A_s^{j-1}\mathscr{V}_{i-1} = \sum_{j=1}^{\vartheta_s} A_s^{j-1}\mathscr{V}_{i-1}. \tag{7}$$

On the other hand, a nested subspace is defined by (2). It can be verified directly from (2) that $\mathscr{V}_\mu$ is a sum of various adding terms with each one in the form of $A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1}\mathscr{B}_{i_0}$. Equalities (4) and (7) imply that $0 \leq j_s \leq \vartheta_s - 1, s = 1, \ldots, \mu - 1$. Accordingly, computations according to (2) give rise to $\mathscr{V}_\mu = \mathfrak{M}$, where $\mathfrak{M}$ is given by (3). Since $\mathscr{V} = \mathscr{V}_\mu$, the result then follows from Lemma 2. □

**Remark 1.** The contribution of Theorem 1 consists in providing a simplified geometric characterization for the reachability subspace $\mathscr{T}$, i.e. $\mathscr{T} = \mathfrak{M}$. More specifically, $\mathscr{T}$ was written in Sun and Ge (2005) in the form

$$\mathscr{T} = \sum_{i_0, \ldots, i_{n-1} \in \Lambda}^{j_1, \ldots, j_{n-1} \in \{0, \ldots, n-1\}} A_{i_{n-1}}^{j_{n-1}} \cdots A_{i_1}^{j_1} \mathscr{B}_{i_0}, \tag{8}$$

The difference between (3) and (8) lies in: (i) The number of multiplying matrices in each adding term in (3) is $\mu$ ($\leq n - \mathbb{d}_1 + 1$), which is not greater than $n$, the same kind of number in (8) as $\mu$ in (3). Hence, the number of adding terms in (8) is greatly reduced in (3), especially when $\mu$ is much less than $n$; (ii) The maximum amount of power in association with each multiplying system matrix $A_{i_s}$ in (8) is $n - 1$, which is reduced to $\vartheta_{i_s} - 1$ in (3), $i_s \in \Lambda$. Note that by (5) and (6), $\vartheta_{i_s} \leq \min\{n -$

Download English Version:

https://daneshyari.com/en/article/697208

Download Persian Version:

https://daneshyari.com/article/697208

Daneshyari.com