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Brief paper

On the constrained small-time controllability of linear systems☆

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ABSTRACT

This note presents a necessary and sufficient condition for small-time controllability of a linear system with respect to a cone. This result extends the controllability conditions for linear systems to the case of control and phase constraints.

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1. Introduction

The property of a linear control system to be small-time locally controllable is very important and was studied in a large number of papers. We only mention here that necessary and sufficient controllability conditions for linear systems with controls restricted to a cone have been obtained in Korobov (1980), Bianchini (1983), Brammer (1975), Sussmann (1987b) and Veliov (1988). There are only a few results concerning the controllability of linear systems in the presence of state constraints, despite that control with constraints is increasingly applied in the industry (cf. the motivation of Ko and Bitmead (2007) and the references therein). For example, the directions of expansion of the attainable set of a linear control system in the presence of a phase constraint are studied in Krastanov and Veliov (1992).

We consider the following linear control system

$$\dot{x}(t) \in Ax(t) + U, \tag{1}$$

in the presence of the phase constraint $\langle l, x(t)\rangle \geq 0$, where A is an $(n \times n)$-matrix, the vector $l \neq 0$ belongs to $\mathbf{R}^n$, the phase variable $x(t) \in \mathbf{R}^n$ and $U \subseteq \mathbf{R}^n$ is a closed convex cone in $\mathbf{R}^n$ (a subset U of a real vector space is a convex cone if $\alpha_1 u_1 + \alpha_2 u_2 \in U$ for each u_1 and u_2 of U and for each $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$).

By definition, a trajectory of the control system (1) defined on $[0, T]$ is any absolutely continuous function $x : [0, T] \to \mathbf{R}^n$ satisfying the phase constraint and inclusion (1) for almost every $t \in [0, T]$.

In the present paper we study the small-time controllability of (1). We set $M_0^+ := \{x \in \mathbf{R}^n : \langle l, x\rangle \geq 0\}$. Denote by $A(x, T)$ the attainable set of system (1) at time T starting from the point x at $t = 0$, i.e. $A(x, T) = \{y \in \mathbf{R}^n$: there exists a trajectory $x : [0, T] \to M_0^+$ of (1) such that $x(0) = x$ and $x(T) = y\}$.

Definition 1. The control system (1) is called small-time controllable (STC) if $A(0, T) = M_0^+$ for each $T > 0$.

Remark 2. We mention that due to the linearity of control system (1) and due to the cone constraints on the control, the set $A(0, T)$ is a closed convex cone. Hence STC is equivalent to small-time local controllability (this property is defined by replacing the requirement for $A(0, T)$ with "$A(0, T)$ contains the origin in its relative interior with respect to M_0^+").

The next proposition is proved in Krastanov and Veliov (1992). It shows the importance of the existence of a vector $\tilde{u} \in U$ such that $\langle l, \tilde{u}\rangle < 0$. If such a vector does not exists, then system (1) is STC only under rather specific conditions which are easily checkable, as it is seen below:

Proposition 3. *We assume that the control system (1) is STC and that $\langle l, u\rangle \geq 0$ for each $u \in U$. Then the following conditions are fulfilled:*

(a) *l is an eigenvector of the matrix A^{T} (the notation $^{\mathrm{T}}$ means transposition);*
(b) *there exists a vector $\bar{u} \in U$ such that $\langle l, \bar{u}\rangle > 0$;*
(c) *the control system*

$$\dot{x}(t) \in Ax(t) + U \cap M_0, \quad x(0) = 0, \tag{2}$$

is STC when it is considered on

$$M_0 := \{x \in \mathbf{R}^n : \langle l, x\rangle = 0\}.$$

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Proposition 3 implies that $\max_{u\in U}|\langle l,u\rangle| > 0$ is a necessary condition for the STC of (1). This condition, however, is not fulfilled by many control systems of interest. For that reason we study the small-time controllability of (1) with respect to a suitably defined cone. The main result of this paper is a necessary and sufficient condition. The proof of this condition is formally similar to the one developed in Krastanov and Veliov (2005), Veliov (1988) and Veliov and Krastanov (1986). However, in essence, it is very different. To prove the sufficiency, we use suitable control variations. Their construction is not a trivial task because the origin is a boundary point of the state constraint Γ and all admissible velocities at the origin are tangent to Γ. To prove the necessity, we study the projection of the reachable set $A(0,T)$, $T > 0$, on a suitably defined linear subspace.

2. Small-time controllability with respect to a closed convex cone

Using the Cauchy formula to compute the solution of a linear ODE, one can easily obtain the following two properties of the attainable set:

(A1) $A(0,t_1) \subseteq A(0,t_2)$ for every two nonnegative numbers t_1 and t_2 with $t_1 \le t_2$;
(A2) the set $A(0,t)$ is a convex cone for each nonnegative number t.

Definition 4. Let C be a closed convex cone in $\mathbf{R}^n$. The control system (1) is called small-time controllable (STC) with respect to the cone C if the cone C is a subset of the attainable set $A(0,T)$ for every $T > 0$.

Following some of the ideas proposed in Hermes (1978), Krastanov and Veliov (2005), Sussmann (1987a), Veliov (1988) and Veliov and Krastanov (1986), we define the following set E_C^+ of tangent vectors:

Definition 5. Let C be a convex closed cone in $\mathbf{R}^n$. A vector p in C is a tangent vector to the attainable set at the origin with respect to the cone C if there exist two positive constants c and $w > 1$, and two continuous functions $\varphi : \mathbf{R}_+ \times [0,1] \to C$ and $\rho : [0,1] \to \mathbf{R}_+$ such that ρ is increasing, $\rho(0) = 0$ and for each $t \in [0,1]$ and each $\alpha > 0$ the following relation holds true

$$t\alpha p + \varphi(\alpha,t) \in A(0,\rho(t)) \quad \text{with } \|\varphi(\alpha,t)\| \le c\alpha t^w.$$

We denote by E_C^+ the set of all tangent vectors to the attainable set at the origin with respect to the cone C. In particular, we denote the set $E_{\mathbf{R}^n}^+$ by E^+.

Lemma 6. *Let L be a linear subspace of $\mathbf{R}^n$ and let C be the cone generated by L and the nonzero vector p. We assume that*

(a) *the vector p belongs to the set E_C^+;*
(b) *the vectors $p_1,\ldots,p_k$ belong to the set E_L^+ and*

$$0 \in \text{relint}\ \text{co}\{p_i,\ i = 1,\ldots,k\}, \tag{3}$$

where "relint" and "co" mean the relative interior with respect to L and the convex hull, respectively.

Then the linear control system (1) is STC with respect to the cone C.

The following lemmas provide constructions of elements of the set E_C^+.

Lemma 7. *Let C be a convex closed cone in $\mathbf{R}^n$. Then the set of tangent vectors E_C^+ is also a convex cone.*

Lemma 8. *Let C be a closed convex cone in $\mathbf{R}^n$ and $B : \mathbf{R}^n \to \mathbf{R}^n$ be a linear operator with $B(C) \subseteq C$. We assume that*

(a) *the vectors p and $-p$ belong to the set E_C^+;*
(b) *$e^{B\tau}(A(0,t)\bigcap C) \subseteq A(0,t+\tau)$ for each positive numbers τ and t in $[0,1]$.*

Then the vectors Bp and $-Bp$ belong to the set E_C^+.

The proofs of Lemmas 6–9 are given in the Appendix.

3. A necessary and sufficient controllability condition

We consider the system (1) under the following

Main assumption. There exists a positive integer k such that

$$\langle l, A^k b\rangle > 0 \quad \text{for some } b \in U \text{ with } -b \in U \text{ and } \langle l, A^i u\rangle = 0,$$

for each $u \in U$ and for each $i = 0, 1, \ldots, k-1$.

Under this assumption Proposition 3 implies that the control system (1) is not STC. Nevertheless, it is small-time controllable with respect to a suitable defined cone K. First, we set

$$V := \left\{ u - \frac{\langle l, A^k u\rangle}{\langle l, A^k b\rangle} b : u \in U \right\}.$$

It can be verified that V is a convex closed cone contained in the linear subspace

$$M_k := \left\{ x \in \mathbf{R}^n : \langle l,x\rangle = \cdots = \langle l, A^k x\rangle = 0 \right\}.$$

Next, we define the linear operator P as follows:

$$Px = Ax - \frac{\langle l, A^{k+1}x\rangle}{\langle l, A^k b\rangle} b, \quad x \in \mathbf{R}^n.$$

It can be verified that the linear subspace M_k is invariant with respect to the action of P. Also, the vector $P^{k+1}b$ belongs to M_k. Finally, we denote by K the cone generated by M_k and the vector $P^k b$, and set $C_b := \{\alpha b : \alpha \in \mathbf{R}\}$. Then for each $x \in \mathbf{R}^n$ and $u \in U$, $Ax + u \in Px + C_b + V$. Also, for each $\alpha b \in C_b$ and $v \in V$, $Px + \alpha b + v \in Ax + U$. Hence, instead of studying the STC property of control system (1), one can study the same property for the system:

$$\dot{x}(t) \in Px(t) + C_b + V, \quad x(0) = 0 \in \mathbf{R}^n, \tag{4}$$

under the same phase constraint.

Lemma 9. *The following assertions hold true*

(a) *the cone V is a subset of the set $E_{M_k}^+$;*
(b) *the vector $P^{k+1}b$ belongs to the set $E_{M_k}^+$;*
(c) *the vector $P^k b$ belong to the set E_K^+.*

Below $\text{Rec}(V)$ stands for the maximal subspace contained in the convex cone $V \subset \mathbf{R}^n$, and $\text{Inv}(S)$ is the minimal linear subspace of $\mathbf{R}^n$ that contains the set S and is invariant with respect to the linear operator P.

Introduce successively the subspaces

$$L_1 = \left(\text{Inv}\left(\text{Rec}\left(\text{co}\left(V \cup \{P^{k+1}b\}\right)\right)\right)\right),$$

$$L_2 = \left(\text{Inv}\left(\text{Rec}\left(\text{co}\left(V \cup \{P^{k+1}b\} \cup L_1\right)\right)\right)\right),$$

$$\ldots$$

$$L_{s+1} = \left(\text{Inv}\left(\text{Rec}\left(\text{co}\left(V \cup \{P^{k+1}b\} \cup L_s\right)\right)\right)\right),$$

$$\ldots.$$

Theorem 10. *Consider the control system (1) in the presence of the phase constraint $\langle l,x\rangle \ge 0$. Let us assume that the Main assumption holds true. Then control system (1) is STC with respect to the cone K if and only if*

$$M_k = L_m \quad \text{for some } m \le n-k-2. \tag{5}$$

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