Automatica 44 (2008) 2398–2402

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Root-mean-square gains of switched linear systems: A variational approach ☆

Michael Margaliot [a,*], João P. Hespanha [b]

[a] School of Electrical Eng.–Systems, Tel Aviv University, 69978, Israel

[b] Department of Electrical and Computer Eng., University of California, Santa Barbara, CA 93106-9560, USA

ARTICLE INFO

Article history:
Received 21 February 2007
Received in revised form
17 December 2007
Accepted 24 January 2008
Available online 5 May 2008

Keywords:
Switched and hybrid systems
Bilinear control systems
Optimal control
Maximum principle
Algebraic Riccati equation
Differential Riccati equation
Hamilton–Jacobi–Bellman equation

ABSTRACT

We consider the problem of computing the root-mean-square (RMS) gain of switched linear systems. We develop a new approach which is based on an attempt to characterize the "worst-case" switching law (WCSL), that is, the switching law that yields the maximal possible gain. Our main result provides a sufficient condition guaranteeing that the WCSL can be characterized explicitly using the differential Riccati equations (DREs) corresponding to the linear subsystems. This condition automatically holds for first-order SISO systems, so we obtain a complete solution to the RMS gain problem in this case.

1. Introduction

Consider the switched linear system

$$\dot{\boldsymbol{x}} = A_{\sigma(t)}\boldsymbol{x} + B_{\sigma(t)}\boldsymbol{u}, \qquad \boldsymbol{y} = C_{\sigma(t)}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{y} \in \mathbb{R}^k$, and the *switching signal* $\sigma : \mathbb{R}_+ \to \{1, 2, \ldots, l\}$ is a piecewise constant function specifying, at each time instant t, the index of the currently active system. Roughly speaking, (1) models a system that can switch between the l linear sub-systems:

$$\dot{\boldsymbol{x}} = A_i\boldsymbol{x} + B_i\boldsymbol{u}, \qquad \boldsymbol{y} = C_i\boldsymbol{x}, \tag{2}$$

for $i = 1, \ldots, l$. Note that we consider subsystems with no direct input-to-output term. To avoid some technical difficulties, we assume throughout that each linear subsystem is a minimal realization.

Let $\mathcal{S}$ denote the set of all piecewise constant switching laws. Many important problems in the analysis and design of switched systems can be phrased as follows.

Problem 1. Given $\mathcal{S}' \subseteq \mathcal{S}$ and a property P of dynamic systems, determine whether the switched system (1) satisfies property P for **every** $\sigma \in \mathcal{S}'$.

For example, when $\boldsymbol{u} \equiv 0$, P is the property of asymptotic stability of the origin, and $\mathcal{S}' = \mathcal{S}$, Problem 1 specializes into the following problem.

Problem 2 (*Liberzon, 2003; Shorten, Wirth, Mason, Wulff, & King, 2007*). Is the switched system (1) asymptotically stable under arbitrary switching laws?

Solving Problem 1 is difficult for two reasons. First, the set $\mathcal{S}'$ is usually huge, so exhaustively checking the system's behavior for each $\sigma \in \mathcal{S}'$ is impossible. Second, it is entirely possible that each of the subsystems satisfies property P, yet the switched system admits a solution that does *not* satisfy property P. Thus, merely checking the behaviors of the subsystems is not enough.

A general approach for addressing Problem 1 is based on studying the "worst-case" scenario. We say that $\tilde{\sigma} \in \mathcal{S}'$ is the *worst-case switching law* (WCSL) in $\mathcal{S}'$, with respect to property P, if the following condition holds: if the switched system satisfies property P for $\tilde{\sigma}$, then it satisfies property P for *any* $\sigma \in \mathcal{S}'$. Thus, the analysis of property P under *arbitrary* switching signals from $\mathcal{S}'$ can be reduced to analyzing the behavior of the switched system for the *specific* switching signal $\tilde{\sigma}$.

The WCSL for Problem 2 (that is, the "most destabilizing" switching law) can be characterized using variational principles

☆ An abridged version of this manuscript was presented at the IEEE Conf. on Decision and Control (CDC'07). Prof. Hespanha's research was supported by NSF Grants ECS-0242798 and CCR-0311084. This paper was recommended for publication in revised form by Associate Editor Maria Elena Valcher under the direction of Editor Roberto Tempo.

* Corresponding author. Tel.: +972 3 640 7768; fax: +972 3 640 5027.
E-mail addresses: michaelm@eng.tau.ac.il (M. Margaliot), hespanha@ece.ucsb.edu (J.P. Hespanha).

0005-1098/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2008.01.026

(see the survey paper Margaliot (2006)). This idea originated in the pioneering work of Pyatnitskii on the celebrated *absolute stability problem* (Pyatnitskii, 1970, 1971) (see also Barabanov (2005), Margaliot and Branicky (in press), Pyatnitskiy and Rapoport (1996) and Rapoport (1996)). The basic idea is to embed the switched system in a more general bilinear control system.[1] Then, the "most destabilizing" switching law can be characterized as the solution to a suitable optimal control problem. For second-order systems, this problem can be explicitly solved using the generalized first integrals of the subsystems (Holcman & Margaliot, 2003; Margaliot & Langholz, 2003).

In this paper, we use a similar approach for studying the root-mean-square (RMS) gain problem. Our main result shows that if a certain condition holds, then the WCSL can be obtained by switching between the l differential Riccati equations (DREs) corresponding to the l linear subsystems. This condition automatically holds for first-order SISO systems, so we obtain a complete solution to the RMS gain problem for this case.

We use standard notation. Vectors are denoted by boldface letters, and the transpose of a vector $\boldsymbol{v}$ is denoted by $\boldsymbol{v}'$. Matrices are denoted by capital letters. If P, Q are two symmetric matrices, then $P > Q$ means that $P - Q$ is positive-definite.

2. Preliminaries

In this section, we describe some known results on the RMS gain of (non-switched) linear systems and the associated Riccati equations.

Consider the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{y} = C\boldsymbol{x}, \tag{3}$$

with (A, B) controllable and (A, C) observable. For $T \in (0, +\infty]$, let $\mathcal{L}_{2,T}$ denote the set of functions $\boldsymbol{f}(\cdot)$ such that $\|\boldsymbol{f}\|_{2,T} := \left(\int_0^T \boldsymbol{f}'(t)\boldsymbol{f}(t)\mathrm{d}t\right)^{1/2} < \infty$. The RMS gain over $[0, T]$ of (3) is defined by $g(T) := \inf\{\gamma \geq 0 : \|\boldsymbol{y}\|_{2,T} \leq \gamma\|\boldsymbol{u}\|_{2,T},\ \forall \boldsymbol{u} \in \mathcal{L}_{2,T}\}$, where $\boldsymbol{y}$ is the output of (3) corresponding to $\boldsymbol{u}$ with $\boldsymbol{x}(0) = \boldsymbol{0}$.

It is well-known that $g(\infty) = \|C(sI-A)^{-1}B\|_\infty$, where $\|Q(s)\|_\infty := \sup_{Re(s)\geq 0} \|Q(s)\|$ is the $\mathbb{H}_\infty$ norm of the transfer matrix $Q(s)$ (Green & Limebeer, 1995).

The RMS gain can also be computed by solving a suitable Riccati equation (Zhou, Doyle, & Glover, 1996). This equation arises because the Hamilton–Jacobi–Bellman equation, characterizing a suitable dissipation function (van der Schaft, 1992), admits a solution with a quadratic form for linear systems. Fix $T \in (0, \infty)$, a symmetric matrix $P_T \in \mathbb{R}^{n\times n}$, and consider the DRE:

$$\dot{P}(t) = -S_\gamma(P(t)), \qquad P(T) = P_T, \tag{4}$$

where $S_\gamma(P) := PA + A'P + C'C + \gamma^{-2}PBB'P$. Let $I(P_T) \subseteq [0, T]$ denote the maximal time interval for which the solution $P(t)$ exists.

Theorem 1 (*Hespanha, 2002*). $g(T) = \inf\{\gamma \geq 0 : I(0) = [0, T]\}$.

The next result, which is a special case of Abou-Kandil, Freiling, Ionescu, and Jank (2003, Theorem. 4.1.8), provides a sufficient condition for $P(t)$ to be monotonic.

Theorem 2. *If* $\dot{P}(T) \leq 0$ *then* $\dot{P}(t) \leq 0$, *for all* $t \in I(P_T)$.

The *algebraic Riccati equation* (ARE) associated with the DRE (4) is:

$$S_\gamma(P) = 0. \tag{5}$$

Theorem 3 (*Hespanha, 2002*). *Consider the linear system (3), where A is Hurwitz. Fix $\gamma > 0$ and denote $R := \gamma^{-2}BB'$. If $\gamma > g(\infty)$ then the ARE (5) admits symmetric and positive definite solutions P^-, $P^+ \in \mathbb{R}^{n\times n}$, referred to as the* stabilizing *and* antistabilizing *solutions, respectively, such that $A+RP^-$ and $-(A+RP^+)$ are Hurwitz. Moreover, $P^+ > P^-$.*

It is possible to express the solution to the DRE (4) using P^- and P^+.

Theorem 4 (*Hespanha, 2002*). *Suppose that the conditions of Theorem 3 hold, and that $P_T - P^+$ is nonsingular. Define $Q := A+RP^+$, and*

$$\Lambda(t) := \mathrm{e}^{Q(t-T)}\left((P_T - P^+)^{-1} + (P^+ - P^-)^{-1}\right)\mathrm{e}^{Q'(t-T)} - (P^+ - P^-)^{-1}, \quad t \leq T.$$

(1) *The solution to (4) is $P(t) = P^+ + (\Lambda(t))^{-1}$, for all $t \in I$, where $I \subset (-\infty, T]$ is an interval on which Λ is nonsingular.*
(2) *If $P_T - P^+ < 0$, then $P(t)$ exists for all $t \leq T$ and $\lim_{t\downarrow-\infty} P(t) = P^-$.*
(3) *If $P_T - P^+ \not\leq 0$, the solution $P(t)$ has a finite escape time,[2] that is, there exist $\tau \in (-\infty, T)$ and $\boldsymbol{z} \in \mathbb{R}^n$ such that $\lim_{t\downarrow\tau} \boldsymbol{z}'P(t)\boldsymbol{z} = +\infty$.*

Theorem 4 implies in particular that $g(T) \leq g(\infty)$ for all T. Indeed, seeking a contradiction, suppose that there exists a time T such that $g(T) > g(\infty)$. Fix γ such that $g(\infty) < \gamma < g(T)$. Theorem 3 implies that for this γ the ARE admits solutions $P^-, P^+ > 0$. Part (2) of Theorem 4 implies that the solution $P(t)$ to the DRE, with $P_T = 0$, exists for all $t \leq T$. Theorem 1 now yields $g(T) \leq \gamma$, which is a contradiction.

3. RMS gain of switched systems

The RMS gain of (1), over some set of switching signals $\mathcal{S}' \subseteq \mathcal{S}$, is defined by

$$g_{\mathcal{S}'}(T) := \inf_{\gamma\geq 0}\{\|\boldsymbol{y}\|_{2,T} \leq \gamma\|\boldsymbol{u}\|_{2,T}, \forall \boldsymbol{u} \in \mathcal{L}_{2,T}, \forall \sigma \in \mathcal{S}'\},$$

where $\boldsymbol{y}$ is the solution to (1) corresponding to $\boldsymbol{u}$, σ, with $\boldsymbol{x}(0) = \boldsymbol{0}$. This can be interpreted as the "worst-case" energy amplification gain for the switched system, over all possible input and switching signals in $\mathcal{S}'$. Computing $g_{\mathcal{S}'}(T)$ is an important open problem in the design and analysis of switched systems (Hespanha, 2004). In particular, calculating induced gains is the first step toward the application of robust control techniques to switched systems (Lin, Zhai, & Antsaklis, 2006; Zhai, Lin, Kim, Imae, & Kobayashi, 2005; Zhao & Hill, 2005).

By definition, $g_{\mathcal{S}}(T) \geq \max_{1\leq i\leq l}\{g_i(T)\}$, where $g_i(T)$ is the RMS gain of the ith linear subsystem. Consider the switched system (1) with $\boldsymbol{u} \equiv \boldsymbol{0}$. It is well-known that global asymptotic stability of the individual linear subsystems is necessary, but not sufficient for global asymptotic stability of the switched system for every $\sigma \in \mathcal{S}$ (Liberzon, 2003). We assume from here on that the switched system with $\boldsymbol{u} \equiv \boldsymbol{0}$ is *globally uniformly asymptotically stable* (GUAS), that is, there exist $\lambda_1, \lambda_2 > 0$ such that $\|\boldsymbol{x}(t)\| \leq \lambda_1\|\boldsymbol{x}(0)\|\mathrm{e}^{-\lambda_2 t}$, for all $t \geq 0$, $\sigma \in \mathcal{S}$, and $\boldsymbol{x}(0) \in \mathbb{R}^n$. In particular, this implies of course that A_i, $i = 1, \ldots, l$, are all Hurwitz. The next example, adapted from Hespanha (2003), shows that even in this case, the RMS gain of the switched system can be very different from that of the subsystems.

[1] For a recent and comprehensive presentation of bilinear systems, see Elliott (in press).

[2] See Abou-Kandil et al. (2003) and Bolzern, Colaneri, and De Nicolao (1997) for some related considerations.