Automatica 45 (2009) 1354–1358

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Finite-time stability of linear time-varying systems with jumps[☆]

Francesco Amato[a], Roberto Ambrosino[b], Marco Ariola[b,*], Carlo Cosentino[a]

[a] *School of Computer and Biomedical Engineering, Università degli Studi Magna Græcia di Catanzaro, Viale Europa, Campus di Germaneto, 88100 Catanzaro, Italy*

[b] *Dipartimento per le Tecnologie, Università degli Studi di Napoli Parthenope, Centro Direzionale, Isola C4, 80143, Napoli, Italy*

A R T I C L E I N F O

Article history:
Received 14 November 2007
Received in revised form
12 September 2008
Accepted 29 December 2008
Available online 28 February 2009

Keywords:
Finite-time stability
Sampled-data systems
Jump linear systems

A B S T R A C T

This paper deals with the finite-time stability problem for continuous-time linear time-varying systems with finite jumps. This class of systems arises in many practical applications and includes, as particular cases, impulsive systems and sampled-data control systems. The paper provides a necessary and sufficient condition for finite-time stability, requiring a test on the state transition matrix of the system under consideration, and a sufficient condition involving two coupled differential–difference linear matrix inequalities. The sufficient condition turns out to be more efficient from the computational point of view. Some examples illustrate the effectiveness of the proposed approach.

1. Introduction

The concept of finite-time stability (FTS) dates back to the fifties, when it was introduced in the Russian literature (Kamenkov, 1953; Lebedev, 1954a,b); later during the sixties this concept appeared in the western control literature (Dorato, 1961; Weiss & Infante, 1967). A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa. While LAS deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturations or the excitation of nonlinear dynamics) during the transients.

In Amato, Ariola, and Cosentino (2006) and Amato, Ariola, and Dorato (2001) a sufficient condition for FTS and finite-time stabilization of continuous-time linear time-invariant systems was provided, by using an approach based on the Lyapunov function theory; such a condition requires the solution of a feasibility problem involving Linear Matrix Inequalities (LMIs). A different approach, which is reminiscent of optimal control techniques and is also applicable to linear time-varying systems, has been proposed in Amato, Ariola, Carbone, and Cosentino (2006) and Amato, Ariola, and Cosentino (2005). In the time-invariant case, the main result of Amato et al. (2006) turns out to be less conservative than the condition provided in Amato et al. (2001), but it is computationally more demanding since it involves the solution of a Differential Linear Matrix Inequality (DLMI).

In this paper we consider the class of linear time-varying systems with finite state jumps. The concept of linear systems with jumps was firstly proposed in Sun, Nagpal, and Khargonekar (1993). Roughly speaking, such systems are linear continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time. Obviously, the class of linear systems with jumps contains continuous-time linear systems, but, as a matter of fact, it captures many other cases of practical interest in engineering applications, e.g. impulsive (Yang, 2001), hybrid and sampled-data control systems. In particular (see Sun et al. (1993)), systems with jumps were introduced as a suitable framework for representing closed-loop sampled-data systems in which the inter-sample behavior is of interest.

This work extends the approach proposed in Amato et al. (2006) and Amato et al. (2005) to derive the main results for FTS analysis of linear systems with jumps. The first contribution of the paper is a necessary and sufficient condition for FTS. It requires the computation of the state transition matrix of the system, which represents a numerically hard problem. In order to tackle such a problem, we also provide a sufficient condition for FTS, which requires the solution of two coupled differential–difference Lyapunov inequalities. The Lyapunov inequalities can be turned into differential–difference linear matrix inequalities (D/DLMIs),

[☆] This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Vladimir O. Nikiforov under the direction of Editor André L. Tits.

[*] Corresponding author. Tel.: +39 081 5476719; fax: +39 081 5476777.
E-mail address: ariola@uniparthenope.it (M. Ariola).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2008.12.016

which can be efficiently solved by many off-the-shelf numerical algorithms.

The paper is organized as follows. In Section 2 the definition of FTS for a linear system with jumps is precisely stated and the main problems we will deal with are defined. In Section 3 the analysis conditions are given and then they are extended to the case when the jump time instants are unknown. In Section 4 these conditions are exploited to analyze the closed-loop system resulting from the interconnection of a continuous-time system with a sampled-data controller. Finally some conclusions are drawn in Section 5.

Notations. By $\mathcal{L}^2_{[0,T]}$ (l^2_r) we denote the set of square integrable (summable) vector-valued functions defined over the interval $[0, T]$ (over the set $\{1, 2, \ldots, r\}$).

2. Problem statement

Let us consider the continuous-time linear time-varying system described by

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0, \quad x(0) = x_0, \tag{1}$$

where $A(\cdot) : t \in [0, +\infty) \mapsto \mathbb{R}^{n\times n}$ is continuous. We assume that the evolution of the state vector $x(t)$ may be right discontinuous at the time instants $t_k > 0$, where the state may exhibit a finite jump from $x(t_k)$ to $x(t_k^+) \neq x(t_k)$, according to the decision of a "jump generator system". Since $A(\cdot)$ is continuous, $x(\cdot)$ cannot have a finite escape time. Therefore $x(\cdot)$ turns out to be a piecewise continuous function, i.e. in each compact interval of $[0, +\infty)$ it has a finite number of discontinuities and the right and left limits at the discontinuity points are both finite.

The proposed formalism captures many cases of practical interest. For example if $\left(t_k, x(t_k^+)\right)$ is generated through an impulsive input entering the system, we reobtain the class of impulsive control systems (Yang, 2001), whereas if $\left(t_k, x(t_k^+)\right)$ is computed according to a given algorithm (for example it is the output of a discrete event system (Boel & Stremersch, 2000)) the system falls into the category of hybrid control systems (Antsaklis, 1995).

According to Sun et al. (1993), we initially consider the case when the jump time instants are assigned, say t_k, and the state jump is computed as the output of a discrete-time system, described by the following difference equation

$$x(t_k^+) = A_d(t_k)x(t_k), \quad k = 1, 2, \ldots, \tag{2}$$

where $A_d(\cdot) : t_k \mapsto \mathbb{R}^{n\times n}$. Note that, according to Sun et al. (1993), sampled-data systems can be described through (1) and (2) (see also Section 4). As we will show later, our approach can also be extended to the case when the jump time instants are unknown, though the conditions for FTS in that case turn out to be more conservative.

In this paper we are interested in the behavior of system (1)–(2) within a finite time interval $[0, T]$, therefore we denote by $t_1, \ldots, t_r$, $r \in \mathbb{N}^+$, the jump time instants belonging to such an interval. The solution of system (1)–(2) in $[0, T]$ is given by

$$x(t) = \Phi(t, 0)x_0, \quad t \in [0, T], \tag{3}$$

where the matrix function $\Phi(t, \tau)$ is the state transition matrix of system (1)–(2). The transition matrix turns out to be piecewise continuous with possible right discontinuities at the time instants t_k, $k = 1, 2, \ldots, r$. In the first interval $\Phi(t, \tau)$ satisfies the following matrix differential equation

$$\frac{\partial}{\partial t}\Phi(t, 0) = A(t)\Phi(t, 0), \quad t \in [0, t_1], \quad \Phi(0, 0) = I;$$

in the following intervals, for $k = 1, 2, \ldots, r-1$,

$$\frac{\partial}{\partial t}\Phi(t, t_k^+) = A(t)\Phi(t, t_k^+), \quad t \in]t_k, t_{k+1}],$$
$$\Phi(t_k^+, t_k^+) = A_d(t_k)\Phi(t_k, t_{k-1}^+),$$

where $t_0^+ = t_0 := 0$ (obviously at $t_0 = 0$ there is no discontinuity). Finally in the last interval we have

$$\frac{\partial}{\partial t}\Phi(t, t_r^+) = A(t)\Phi(t, t_r^+), \quad t \in]t_r, T]$$
$$\Phi(t_r^+, t_r^+) = A_d(t_r)\Phi(t_r, t_{r-1}^+).$$

In the following we extend the definition of FTS (Amato et al., 2001; Weiss & Infante, 1967), to the class of linear systems with jumps in the form (1)–(2).

Definition 1 (*FTS of Linear Systems with Jumps*). Given a positive number T, a positive definite matrix R, a positive definite matrix-valued function $\Gamma(\cdot)$ defined over $[0, T]$, with $\Gamma(0) < R$, system (1)–(2) is said to be finite-time stable with respect to $\left(T, R, \Gamma(\cdot)\right)$, if

$$x_0^{\mathrm{T}}Rx_0 \leq 1 \Rightarrow x(t)^{\mathrm{T}}\Gamma(t)x(t) < 1 \quad \forall t \in [0, T]. \tag{4}$$

Remark 2. The definition can be interpreted in terms of ellipsoidal domains. The set defined by $x_0^{\mathrm{T}}Rx_0 \leq 1$ contains all the admissible initial states. The inequality $x(t)^{\mathrm{T}}\Gamma(t)x(t) < 1$, instead, defines a time-varying ellipsoid that bounds the state trajectory over the interval $[0, T]$.

Given a piecewise continuous vector-valued function $z(\cdot) \in \mathcal{L}^2_{[0,T]}$, with right discontinuities at the points $t_1, \ldots, t_r$, we can define three norms: the first is the classical $\mathcal{L}^2$-norm, denoted by $\|z\|_{2,L}$. Then, note that the function $z(\cdot)$ univocally defines the sequence $\{z(t_k)\}_{k=1,2,\ldots,r} \in l^2_r$ (we recall that $z(t_k)$ represents the left limit of $z(\cdot)$ in t_k and that $z(\cdot)$ is assumed to be left continuous in t_k); therefore we can also consider the classical l^2-norm of the sequence $\{z(t_k)\}$, denoted by $\|z\|_{2,l}$. Notice that $\|z\|_{2,l}$ turns out to be a semi-norm for the signal $z(\cdot)$.

Finally, we can think of $z(\cdot)$ as the composition of two signals, one belonging to $\mathcal{L}^2$ and the other to l^2, therefore defining a "mixed norm" over $\mathcal{L}^2 \oplus l^2$

$$\|z\|_{2,m} := \left[\|z\|^2_{2,L} + \|z\|^2_{2,l}\right]^{1/2}. \tag{5}$$

It is simple to recognize that the mixed norm (5) is actually a norm for $\mathcal{L}^2 \oplus l^2$.

3. Main results

The following theorem provides a necessary and sufficient condition for the FTS of system (1)–(2) involving the state transition matrix defined in the previous section. The proof can be readily obtained by using the properties of the transition matrix (see also Amato et al. (2005)).

Theorem 3. *System (1)–(2) is FTS with respect to $(T, R, \Gamma(\cdot))$* iff *for all $t \in [0, T]$*

$$\Phi(t, 0)^{\mathrm{T}}\Gamma(t)\Phi(t, 0) < R. \tag{6}$$

The condition in Theorem 3 may be difficult to apply, unless we are in the time-invariant case, because it requires the computation of the transition matrix. For this reason, we provide an alternative condition for FTS which involves two coupled differential–difference Lyapunov inequalities. The following lemma is the key result.