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Optimal mean–variance control for discrete-time linear systems with Markovian jumps and multiplicative noises[☆]

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ABSTRACT

In this paper, we consider the stochastic optimal control problem of discrete-time linear systems subject to Markov jumps and multiplicative noises under two criteria. The first one is an unconstrained mean–variance trade-off performance criterion along the time, and the second one is a minimum variance criterion along the time with constraints on the expected output. We present explicit conditions for the existence of an optimal control strategy for the problems, generalizing previous results in the literature. We conclude the paper by presenting a numerical example of a multi-period portfolio selection problem with regime switching in which it is desired to minimize the sum of the variances of the portfolio along the time under the restriction of keeping the expected value of the portfolio greater than some minimum values specified by the investor.

1. Introduction

Recently, there has been an increasing interest in the literature for linear systems subject to Markov jumps and/or multiplicative noises. This kind of models has found many applications in engineering and finance as, for instance, in nuclear fission and heat transfer, population models and immunology, portfolio optimization, etc., and several results related to the control of these systems have already been derived.

Regarding the case of linear systems subject to multiplicative noises on the control and/or state variables, a particular point of interest is that the stochastic linear quadratic (LQ) control problem with indefinite state and/or control weighting matrices may still be well-posed. We recall that a standard assumption in the literature of the LQ control problem in order to guarantee that the problem is well posed is that the state weighing matrix is positive semi-definite and the control weighing matrix is positive definite. Indefinite LQ control problems for linear systems subject to multiplicative noises arise naturally in many practical situations especially in finance (see for instance Chen, Li, & Zhou, 1998; Zhou & Li, 2000). For the continuous-time case, the well-posedness for the indefinite LQ problem was first shown in Chen et al. (1998) and since then several papers have been published on this subject. Sufficient conditions for the well-posedness and several properties of the generalized Riccati equations arising in this indefinite stochastic LQ control problem were studied in Ait Rami, Chen, Moore, and Zhou (2001), Lim and Zhou (1999), Luo and Feng (2004), Wu and Zhou (2002) and Zhu (2005) and through linear matrix inequalities in Ait Rami, Moore, and Zhou (2001) and Ait Rami and Zhou (2000). For the discrete-time case, the optimal control law for the LQ control problem was derived in Beghi and D'Alessandro (1998), considering systems with only control dependent noises. The partially observed case was studied in Moore, Zhou, and Lim (1999). In Ait Rami, Chen, and Zhou (2002), it was shown that the solvability of a generalized difference Riccati equation is necessary and sufficient for the existence of an optimal control for the stochastic indefinite LQ optimal control problem. Regarding the case of linear systems subject to Markov jumps, there has been also a great number of results derived on the control theory for this class of models, and the reader is referred, for instance, to Elliott, Dufour, and Malcolm (2005), the books (Boukas, 2005; Costa, Fragoso, & Marques, 2005) and the references therein for an overview on this subject.

In Li and Zhou (2002), Li, Zhou, and Ait Rami (2003) and Liu, Yin, and Zhou (2005) the authors considered the finite and infinite horizon indefinite LQ regulator problems for continuous-time linear systems with Markov jumps as well as multiplicative noises acting on the parameters of the system. A similar problem

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was treated within a time varying framework and infinite time horizon in Dragan and Morozan (2004). The discrete-time case was studied in Costa and de Paulo (2007, 2008) for the finite and infinite horizon indefinite LQ regulator problems, and in Dombrovskii and Lyashenko (2003) for the LQ control problem with applications to a portfolio optimization problem. Conditions for the mean square exponential stability, stochastic observability and stochastic detectability for this class of models were derived in Dragan and Morozan (2006a,b).

One of the applications of indefinite LQ control problem for linear systems subject to multiplicative noises and possible Markov jumps that recently has found a great deal of interest has to do with the multi-period mean–variance portfolio optimization problem. Mean–variance portfolio optimization is a classical financial problem introduced, for the one-period case, by Markowitz (1959), which paved the foundation for the modern portfolio theory. The main goal is to maximize the expected return for a given level of risk, minimize the risk for a given level of expected return, or minimize a trade-off between the variance of the portfolio and the expected return of the portfolio. There has been nowadays a huge literature on this subject, with some extensions, as can be seen, for instance, in Elton and Gruber (1995), Howe and Rustem (1997), Howe, Rustem, and Selby (1996), Rustem, Becker, and Marty (1995) and Steinbach (2001), among others. The multi-period version of this problem has recently been analyzed in continuous as well as in discrete-time. The continuous-time version of the Markowitz's problem was studied in Zhou and Li (2000), using the indefinite stochastic LQ control theory developed in Chen et al. (1998), with closed-form efficient policies derived, along with an explicit expression of the efficient frontier. The discrete-time version of the mean–variance allocation problem was studied in Li and Ng (2000), and later generalized in Zhu, Li, and Wang (2004) for the risk control over bankruptcy, in Costa and Nabholz (2007) for the case with intermediate restrictions, and in Leippold, Trojani, and Vanini (2004) for the case considering liabilities in the portfolio. More recently a revised mean–variance policy that enables the investor to receive a free cash flow stream during the investment process was derived in Cui, Li, Wang, and Zhu (2010). In Yin and Zhou (2004) and Zhou and Yin (2003) it was presented a Markowitz's mean–variance portfolio selection with regime switching, which was solved through an auxiliary indefinite LQ control problem of a Markov jump with multiplicative noises linear system. The idea of introducing the Markov jumps was that typically the underlying market might have many "modes" or "regimes" that switch among themselves from time to time. The market mode would reflect the state of the underlying economy, the general mood of investors in the market, and so on. Under a different point of view, in Cakmak and Ozeckici (2006), Canakoglu and Ozeckici (2010) and Celikyurt and Ozeckici (2007) the authors considered the discrete-time multi-period mean–variance allocation problem for the case in which the parameters are subject to Markov jumps, following an approach closely related to that in Li and Ng (2000). It should be pointed out that only the final values of the mean and of the variance of the portfolio are considered, which considerably simplifies the problem. By using some numerical procedures based on primal–dual methods, the case in which intermediate values of the mean and of the variance of the portfolio are considered for the bankruptcy control problem was studied in Zhu et al. (2004). Following a similar approach, the case in which the market is subject to Markov jumps was considered in Costa and Araujo (2008) and Wei and Ye (2007).

As remarked above, in the multi-period mean–variance problem usually only the final values of the mean and of the variance are considered, which considerably simplifies the problem. But other criteria could be considered, for instance, one could consider a portfolio optimization problem in which the performance criterion is composed by a linear combination along the time of the trade-off between the variance of the output (which could represent the risk associated to the value of the portfolio or a tracking error between the value of the portfolio and a benchmark) and the expected value of the output (which could represent the expected value of the portfolio or the expected value of the surplus between the portfolio and the benchmark). In this case all the values of the variance and expected value of the output would be involved in the optimization problem, which makes the problem harder to be solved. This kind of problem motivated us to consider in this paper the discrete-time optimal control problem of Markov jump with multiplicative noises linear systems with two performance criteria. The first one, denoted by PU (ν, ξ) and defined in (3), is composed by an unconstrained linear combination along the time of a trade-off (set by the vectors ν and ξ) between the variances and expectations of a scalar output $y(t)$ of the system. The second one, denoted by PC (ν, ϵ) and defined in (4), is composed by a minimization of a linear combination of the variances along the time (weighted by the vector ν) under some restrictions of the minimal value of the output of the system (set by the vector ϵ). As pointed out in Li and Ng (2000), when applying dynamic programming to analytically solve these multi-period mean–variance problems a technical difficulty arises due to the existence of a nonlinear term of the form $E(y(t))^2$ in the objective function. To overcome this difficult we adopt a similar approach as in Li and Ng (2000) and introduce a tractable LQ auxiliary problem in which the control weighing matrix is equal to zero and with the presence of linear terms. This problem, defined in (5) in function of a vector λ, is denoted by $A(\lambda, \nu)$. The optimal control strategy for this problem is explicitly derived from a set of generalized coupled Riccati difference equations and some parameters obtained from some recursive equations. Provided that some conditions are satisfied, the optimal solution of the problems PU (ν, ξ) and PC (ν, ϵ) can then be obtained via the solution of the auxiliary problem $A(\lambda, \nu)$, after setting λ to an appropriate value that depends on ξ for problem PU (ν, ξ), and ϵ for problem PC (ν, ϵ). These results generalize the scalar terminal mean–variance problem considered in Cakmak and Ozeckici (2006), Canakoglu and Ozeckici (2010) and Celikyurt and Ozeckici (2007) for the case with Markov jumps, and in Li and Ng (2000) for the case with no jumps, and also extend the results in Costa and Okimura (2007, 2009) which considers the multidimensional case, but only for the terminal mean–variance. Our approach is to derive explicit conditions for optimality of the control strategy rather than applying numerical primal–dual procedures as in Costa and Araujo (2008), Wei and Ye (2007) and Zhu et al. (2004).

The paper is organized as follows. The notation, some preliminary results and the formulation of the problems PU (ν, ξ) and PC (ν, ϵ) are presented in Section 2. We also present in Theorem 1 an analytical optimal control policy for an auxiliary stochastic LQ control problem $A(\lambda, \nu)$, in terms of a set of interconnected Riccati difference equations and some parameters obtained from some recursive equations. The main results of the paper are in Section 3, where we present in Theorems 2 and 3 explicit conditions for the existence of an optimal control strategy for the problems PU (ν, ξ) and PC (ν, ϵ). Moreover the optimal control strategy for these problems is obtained from the optimal control strategy of the auxiliary problem $A(\lambda, \nu)$ after setting λ to some appropriate values. In Section 4 we present an application of the results to a multi-period portfolio selection problem under regime switching. The paper is concluded in Section 5 with some final comments.

2. Problem formulation and auxiliary problems

Throughout the paper the n-dimensional real Euclidean space will be denoted by $\mathbb{R}^n$ and the linear space of all $m \times n$ real matrices by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, with $\mathbb{B}(\mathbb{R}^n) := \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. We use the standard notation, for $A \in \mathbb{B}(\mathbb{R}^n)$, $A \geq 0$ ($A > 0$ respectively) to denote that the matrix A is positive semi-definite (positive definite), and $\mathrm{tr}(A)$