Available online at www.sciencedirect.com
ScienceDirect
automatica
Automatica 44 (2008) 1045–1054
www.elsevier.com/locate/automatica

Brief paper

Stability and performance recovery within discretized non-linear control systems☆

Guido Herrmann[a,*,1], Sarah K. Spurgeon[b], Christopher Edwards[b]

[a]Dynamics and Control Group, University of Bristol, University Walk, Bristol, BS8 1TR, UK
[b]Control and Instrumentation Group, University of Leicester, University Road, Leicester, LE17 RH, UK

Received 13 January 2003; received in revised form 10 April 2006; accepted 22 August 2007
Available online 21 February 2008

Abstract

In this paper, a closed-loop fast sampling analysis of a discretized (emulated) continuous-time controller in a sampled-data environment is presented. The analysis involves a general weighted version of the L_p-norm. This allows a qualitative and quantitative stability and performance analysis for an emulated controller. Several examples are used to show the relevance of these results for the analysis of sampled-data implementations and the computation of quantitative upper limits on the sampling period to achieve recovery of stability and performance.

Keywords: Output constraint; Override control; Non-smooth Lyapunov function; High-gain component

1. Introduction

Sampled-data systems, combining a continuous-time system with a discrete controller implemented via sample-and-hold elements, have received significant attention from control theorists mostly due to performance related problems which have arisen in practice. The problem of linear sampled-data systems has been largely solved in terms of H_2- and H_∞-system analysis and design (Chen & Francis, 1995; Ichikawa & Katayama, 2001). Linear design methods have been developed which not only account for stability and performance at the sampling instances, but also for appropriate inter-sample behavior according to some measure such as the H_∞-norm or the H_2-norm. In terms of non-linear sampled-data systems, significant effort has been made to try to achieve similar results. The problem has been approached from two points of view: analysis and design of the sampled-data control system based on a discrete-time model valid at the sampling instances (Guilaume, Bastin, & Campion, 1995; Nešić & Laila, 2002; Nešić, Teel, & Kokotović, 1999), while the other approach is to analyze a sampled-data implementation of a continuous-time controller (Grüne, 1999, Herrmann, Spurgeon, & Edwards, 2003a, 2003b; Nešić & Grüne, 2005; Owens, Zheng, & Billings, 1990; Zheng, Owens, & Billings, 1990). Both approaches involve a significant amount of complexity. From an intuitive and practical point of view, the second approach has been of significant interest, as it is often convenient to design a continuous-time controller and then to implement this controller as a sampled-data system after discretization employing a sufficiently high sampling frequency (Laila, Nešić, & Teel, 2002; Nešić & Laila, 2002; Owens et al., 1990; Teel, Nešić, & Kokotović, 1998). This approach is termed emulation.

In this paper, L_p-type approaches will be considered for stability analysis. For linear systems, L_p-results have been the first step for proving a general approach to robust linear sampled-data design. It was a significant result to show that any linear sampled-data system, which is (exponentially) stable for the sampling instances, is also L_p-stable as a sampled-data system for suitably chosen disturbance inputs (Chen & Francis, 1991). This result has been recently extended by Zaccarian, Teel, and Nešić (2003) to exponentially stabilized non-linear

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Andrew R. Teel under the direction of Editor Hassan Khalil.

* Corresponding author. Tel.: +44 117 928 8100/8208; fax: +44 117 929 4423.
E-mail address: g.herrmann@bris.ac.uk (G. Herrmann).

[1] This author was supported by a Marie-Curie-Fellowship, Grant No.: FMBICT983463.

0005-1098/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2007.08.023

sampled-data systems. However, L_p-stability can also be used to prove the recovery of stability or performance of *emulated* controllers for high enough sampling frequency. This is the approach which is pursued in this paper. For performance analysis of an emulated non-linear controller, the usual finite horizon L_p-norm is extended by weighting the investigated function $\mathbf{f} : [0, \infty) \to \mathbb{R}^n$ via a function $w : \mathbb{R} \to \mathbb{R}^+$ and computing the L_p-norm of the product of both functions, $w(\cdot)\mathbf{f}(\cdot)$. This approach of using an $L_{w,p}$-norm has been considered by Mousa, Miller, and Michel (1986) and Desoer and Vidyasagar (1975) for exponentially stable systems with $w(t) = c_1 \mathrm{e}^{c_2 t}$. Here, a more general class of weights $w(t)$ is considered for the non-linear sampled-data system analysis. The weighted $L_{w,p}$-norm approach has the advantage of allowing *qualitative* and *quantitative* analysis of a sampled-data system. Hence, *qualitative* characteristics for a sampled-data system such as stability, asymptotic stability or $L_{w,p}$-gain characteristics can be derived for large enough sampling frequency. For certain systems, it is also possible to obtain with the $L_{w,p}$-norm approach a reasonable prediction of an actual value of the sampling-period which suffices for stability or a given rate of decay when considering asymptotic stability. Note that such a *quantitative* stability or performance analysis of an emulated controller in non-linear sampled-data systems is not readily carried out and the respective results generally tend to be conservative. This has been acknowledged by Owens et al. (1990) and Laila et al. (2002) emphasizing that, in particular, Lipschitz bounds can create conservatism when deriving fast-sampling results. Both Owens et al. (1990) and, in particular, Laila et al. (2002) provide a framework for numerical fast-sampling analysis of non-affine systems with exogenous bounded disturbances. Furthermore, Nešić et al. (1999) and Nešić and Laila (2002) have shown for several examples that the theoretical framework they propose for fast sampling analysis can lead to suitable controller configurations for semi-global stability with different sizes for the regions of attraction and different robustness levels with respect to exogenous disturbances. In this paper, one of these examples is revisited to conduct a stability analysis and an analysis of the degree of decay of the system states. This is possible via the weight from the $L_{w,p}$-approach: In an $L_{w,2}$-framework, a fast-sampling result for a discretized controller can be derived giving explicit values for the sampling-period while the use of Lipschitz bounds is prevented for the chosen examples as much as possible. The simplicity of one chosen non-linear example system allows the computation of the exact stability and performance bounds for the sampling-period. These values are then readily compared to the values computed via the $L_{w,2}$-approach.

2. Preliminaries

Modifying definitions from Desoer and Vidyasagar (1975) and van der Schaft (2000), the notation needed for the *weighted* L_p-norm is considered first: A function $\kappa : [0, \infty) \to [0, \infty)$ is a class $\mathcal{K}^-$-function if it is continuous, non-decreasing and $\kappa(0) = 0$. Note that $\mathcal{K}^-$-functions represent a superset of class $\mathcal{K}$-functions (Khalil, 1992, Definition 3.2), as they do not need to be strictly increasing. This approach may limit conservatism, especially when using upper bounds for the *weighted* L_p-approach. [2]

To allow the case of $p = \infty$, the term *essential supremum* may be defined for a measurable function $\mathbf{f} : [0, \infty) \to \mathbb{R}^n$:

$$\operatorname*{ess\,sup}_{t \geqslant 0} \|\mathbf{f}(t)\| = \inf\{a : \|\mathbf{f}(t)\| \leqslant a \text{ almost for all } t \geqslant 0\},$$

where $\|\cdot\|$ is the Euclidean norm. Define for $1 \leqslant p \leqslant \infty$ and measurable $w : \mathbb{R} \to \mathbb{R}^+$ the normed space $L^n_{w,p}(t_a, t_b)$, $(t_b > t_a \geqslant 0)$, of vector valued measurable functions $\mathbf{f}$ with norm $\|\cdot\|_{w,p,(t_a,t_b)}$

$$L^n_{w,p}(t_a, t_b) = \{\mathbf{f} : (t_a, t_b) \to \mathbb{R}^n \,|\, \|\mathbf{f}\|_{w,p,(t_a,t_b)} < \infty\}, \tag{1}$$

where

$$\|\mathbf{f}\|_{w,p,(t_a,t_b)} = \begin{cases} \sqrt[p]{\int_{t_a}^{t_b} (w(s)\|\mathbf{f}(s)\|)^p \,\mathrm{d}s} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{t_a \leqslant s \leqslant t_b} (w(s)\,\|\mathbf{f}(s)\|) & \text{if } p = \infty. \end{cases} \tag{2}$$

When an interval $(t_a, t_b) = (0, T)$ is considered, the abbreviation $\|\mathbf{f}\|_{w,p,T} \overset{\text{def}}{=} \|\mathbf{f}\|_{w,p,(0,T)}$ is used. The extended space $L^n_{w,pe}$ is given by

$$L^n_{w,pe} = \{\mathbf{f} \in L^n_{w,p}(0, T) \text{ for all } T \geqslant 0\}.$$

Note that the extended space $L^n_{w,pe}$ does not have a norm. A normed subspace $L^n_{w,p} \subset L^n_{w,pe}$, with norm $\|\cdot\|_{w,p}$ may be defined

$$L^n_{w,p} = \{\mathbf{f} : [0, \infty) \to \mathbb{R}^n \,|\, \|\mathbf{f}\|_{w,p} < \infty\},$$
$$\|\mathbf{f}\|_{w,p} = \begin{cases} \sqrt[p]{\int_0^{\infty} (w(s)\|\mathbf{f}(s)\|)^p \,\mathrm{d}s} & \text{for } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{s \geqslant 0}(w(s)\|\mathbf{f}(s)\|) & \text{for } p = \infty. \end{cases} \tag{3}$$

If P is an operator from $L^n_{w,pe}$ to $L^m_{w,pe}$, then the induced norm, $\gamma_{w,p}(P)$, or the $L_{w,p}$-gain of P is given by

$$\gamma_{w,p}(P) = \sup_{T > 0, \mathbf{f} \in L^n_{w,pe}, \|\mathbf{f}\|_{w,p,T} \neq 0} \left(\frac{\|P\mathbf{f}\|_{w,p,T}}{\|\mathbf{f}\|_{w,p,T}} \right)$$
$$\Rightarrow \|P\mathbf{f}\|_{w,p,T} \leqslant \gamma_{w,p}(P)\|\mathbf{f}\|_{w,p,T} + \beta_{w,p}(P), \tag{4}$$

where the scalar $\beta_{w,p}(P)$ is the bias term. It is usual to choose the weight $w(t) = 1$. In this case, the space, the norm and the respective system gain are represented by $L^n_p(0, T)$, L^n_{pe}, $\|\cdot\|_{p,T}$ and $\gamma_p(\cdot)$.

Note that this *weighted* L_p-approach increases the versatility of the L_p-approaches usually used in control analysis, in particular for non-linear control. For L_∞-problems, it is easily understood that an extra assumption $\lim_{t\to\infty}(w(t)) = \infty$ can be used to prove asymptotic convergence while the speed of convergence is determined by the choice of $w(\cdot)$. It will be seen later that this is easily extended to other L_p-problems.

The mathematical convention regarding *the sampling process* is now explained using Francis and Georgiou (1988, p. 827).

[2] Note that $\mathcal{K}^-$-functions have been termed gain functions by Teel (1996). Owing to the fact that $\mathcal{K}^-$-functions are used here in a slightly different context than in Teel (1996), the term $\mathcal{K}^-$-function has been chosen.