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On decentralized negotiation of optimal consensus☆

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Abstract

A consensus problem consists of finding a distributed control strategy that brings the state or output of a group of agents to a common value, a consensus point. In this paper, we propose a negotiation algorithm that computes an optimal consensus point for agents modeled as linear control systems subject to convex input constraints and linear state constraints. By primal decomposition and incremental subgradient methods, it is shown that the algorithm can be implemented such that each agent exchanges only a small amount of information per iteration with its neighbors.

Keywords: Consensus; Optimal trajectory planning; Decentralized optimization; Model predictive control; Convex optimization

1. Introduction

The problem of cooperatively controlling systems composed of a large number of autonomous agents has attracted substantial attention in the control and robotics communities. An interesting instantiation is the consensus problem, see for example the recent survey paper Olfati-Saber, Fax, and Murray (2007) and the references therein. It consists of designing distributed control strategies such that the state or output of a group of agents asymptotically converges to a common value, a *consensus point*. The agents are typically modeled by identical first-order systems with no input constraints.

The main contribution of this paper is a decentralized negotiation algorithm that computes the optimal consensus point for a set of agents modeled as linear control systems. In this paper, the consensus point is a vector that specifies, for example, the position and velocity the agents shall converge to. Our approach allows us to incorporate constraints on the state and the input, which is not easily done for the traditional consensus algorithm, see the discussion in Marden, Arslan, and Shamma (2007). By primal decomposition and incremental subgradient methods we design a decentralized negotiation algorithm, in which each agent performs individual planning of its trajectory and exchanges only a small amount of information per iteration with its neighbors. We show that the cost of reaching the consensus point can be significantly reduced, by letting the agents negotiate to find an optimal or near optimal consensus point, before applying a control signal.

There has been a lot of research activity in this area, and a good starting point for related work is the recent survey paper Olfati-Saber et al. (2007). In particular, if the consensus point is a position and *fixed a priori*[1] (contrary to our approach, where the optimal consensus point is a decision variable) we get a so called rendezvous problem. For this type of problem, much work have been focused on establishing convergence to

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[1] In the consensus literature, the consensus point is typically fixed in the sense that it is computed from the initial conditions using a simple rule, for example, the consensus point could be the average of the starting positions of the agents.

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the fixed consensus point under different communication and visibility conditions, see for example Cortéz, Martínez, and Bullo (2006) and the references therein. Furthermore, optimal control formulations have been used in papers that focus on the convergence of distributed model predictive control (MPC) based strategies to an *a priori* fixed equilibrium point. Dunbar and Murray (2006) propose a decentralized scheme where a given desired equilibrium point is asymptotically reached. The scheme requires coupled subsystems to update and exchange the most recent optimal control trajectories prior to each update step. Stability is guaranteed if each subsystem does not deviate too far from the previous open-loop trajectory. In Keviczky, Borelli, and Balas (2006), the authors propose a strategy where each subsystem solves a finite time optimal control problem. The solution of the problem requires each subsystem to know the neighbors' model, constraints, and state. The strategy also requires the prior knowledge of an overall system equilibrium. Finally, a related distributed optimization problem, focused on formation flight, is considered in Raffard, Tomlin, and Boyd (2004), where the decentralized algorithm is based on dual relaxation. Their approach differs from ours in that they do not consider the consensus problem and that they use dual relaxation instead of primal decomposition.

The outline of the paper is as follows. In Section 2, we formulate the optimal consensus problem. The novel distributed negotiation algorithm is presented in Section 4. Section 5 discusses some control strategies and shows a numerical example. Finally, the paper is concluded in Section 5.

2. Problem formulation

Consider $N > 1$ agents whose dynamics are described by

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t),$$
$$z_i(t) = C_i x_i(t), \quad i = 1, \dots, N, \tag{1}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times p_i}$, and $C_i \in \mathbb{R}^{s \times n_i}$ are observable and controllable. The vector $x_i(0) = x_i^0 \in \mathbb{R}^{n_i}$ is the initial condition and $z_i(t)$ is the performance output. We assume that the inputs are constrained according to

$$(u_i^{\mathrm{T}}(0), u_i^{\mathrm{T}}(1), \dots, u_i^{\mathrm{T}}(T))^{\mathrm{T}} \in \mathscr{U}_i, \quad i = 1, \dots, N, \tag{2}$$

where T is a (fixed) time horizon and $\mathscr{U}_i$ is a convex set. By using standard techniques from MPC, the constraint can encode magnitude and rate constraints on $u_i(t)$, as well as restrictions on linear combinations of the agent states (Maciejowski, 2002, Section 3.2).

Definition 1. Let θ lie in a compact and convex set $\Theta \subset \mathbb{R}^s$. The agents described by (1) reach consensus[2] at time T if

$$z_i(T+k) = \theta \quad \text{for all } k \geqslant 0 \text{ and } i = 1, \dots, N,$$

with

$$u_i(T+k) = u_i(T) \quad \text{for all } k \geqslant 0 \text{ and } i = 1, \dots, N.$$

The objective is to find a consensus point $\theta \in \Theta$ and a sequence of inputs $(u_i^{\mathrm{T}}(0), u_i^{\mathrm{T}}(1), \dots, u_i^{\mathrm{T}}(T))^{\mathrm{T}} \in \mathscr{U}_i$, with $i = 1, \dots, N$, such that consensus is reached at time T. The following cost function is associated to the ith system:

$$V_i(z_i(t), u_i(t-1), \theta) \triangleq (z_i(t) - \theta)^{\mathrm{T}} Q_i (z_i(t) - \theta) + u_i(t-1)^{\mathrm{T}} R_i u_i(t-1), \tag{3}$$

where $Q_i \in \mathbb{R}^{s \times s}$ and $R_i \in \mathbb{R}^{p_i \times p_i}$ are positive definite symmetric matrices that encode the cost of deviating from the consensus point and the cost of control energy for agent i. Let us introduce the following vectors:

$$\mathbf{x}_i \triangleq (x_i^{\mathrm{T}}(1), x_i^{\mathrm{T}}(2), \dots, x_i^{\mathrm{T}}(T+1))^{\mathrm{T}},$$
$$\mathbf{u}_i \triangleq (u_i^{\mathrm{T}}(0), u_i^{\mathrm{T}}(1), \dots, u_i^{\mathrm{T}}(T))^{\mathrm{T}}.$$

Since

$$\mathbf{x}_i = \underbrace{\begin{pmatrix} A_i \\ A_i^2 \\ \vdots \\ A_i^{T+1} \end{pmatrix}}_{\mathbf{E}_i} x_i^0 + \underbrace{\begin{pmatrix} B_i & 0 & \dots & 0 \\ A_i B_i & B_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_i^T B_i & A_i^{T-1} B_i & \dots & B_i \end{pmatrix}}_{\mathbf{F}_i} \mathbf{u}_i,$$

we have $z_i(T) = C_i x_i(T) = \mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta$, where $\mathbf{H}_i \triangleq (0 \ \dots \ C_i \ \ 0)$. We also introduce $\mathbf{U}_i \triangleq A_i^{T+1} - A_i^T$ and $\mathbf{W}_i \triangleq (A_i^T B_i \ \ A_i^{T-1} B_i \ \dots \ B_i) - (A_i^{T-1} B_i \ \ A_i^{T-2} B_i \ \dots \ 0)$. We now formulate the optimization problem,

$$\underset{\mathbf{u}_1,\dots,\mathbf{u}_N,\theta}{\text{minimize}} \quad \sum_{i=1}^{N} \mathbf{V}_i(\mathbf{u}_i, \theta), \tag{4a}$$
$$\text{s.t.} \quad \mathbf{H}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) = \theta, \quad i = 1, \dots, N, \tag{4b}$$
$$\mathbf{U}_i x_i^0 + \mathbf{W}_i \mathbf{u}_i = 0, \quad i = 1, \dots, N, \tag{4c}$$
$$\mathbf{u}_i \in \mathscr{U}_i, \quad i = 1, \dots, N, \tag{4d}$$
$$\theta \in \Theta, \tag{4e}$$

with the cost function

$$\mathbf{V}_i(\mathbf{u}_i, \theta) \triangleq \sum_{t=1}^{T+1} V_i(z_i(t), u_i(t-1), \theta) = (\mathbf{C}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) - \mathbf{1}_{T+1} \otimes \theta)^{\mathrm{T}} \mathbf{Q}_i (\mathbf{C}_i(\mathbf{E}_i x_i^0 + \mathbf{F}_i \mathbf{u}_i) - \mathbf{1}_{T+1} \otimes \theta) + \mathbf{u}_i^{\mathrm{T}} \mathbf{R}_i \mathbf{u}_i,$$

where[3] $\mathbf{Q}_i = \mathbf{I}_{T+1} \otimes Q_i$, $\mathbf{R}_i = \mathbf{I}_{T+1} \otimes R_i$, and $\mathbf{C}_i = \mathbf{I}_{T+1} \otimes C_i$. Notice that the constraint (4b) guarantees consensus at time T and (4c) guarantees that the consensus point is an equilibrium, i.e., $x_i(T) = A_i x_i(T) + B_i u_i(T)$. The constraint (4b) can potentially lead to infeasibility problems, but such problems can be mitigated by replacing the constraint with a penalty term in the objective, penalizing deviations from the consensus point at

[2] By introducing a fixed offset, $\bar{\theta}_i$, one for each agent, it is possible to define a *consensus formation* relative to a global consensus point θ. The condition of consensus formation is that $z_i(T+k) = \theta + \bar{\theta}_i$, for all $k \geqslant 0$ and $i = 1, \dots, N$.

[3] With $\mathbf{1}_{T+1}$ we denote the column vector with $T+1$ ones, with $\mathbf{I}_{T+1}$ the $T+1 \times T+1$ identity matrix, and with $\otimes$ the Kronecker matrix product.