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Asymptotic properties of consensus-type algorithms for networked systems with regime-switching topologies[☆]

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ABSTRACT

This paper is concerned with asymptotic properties of consensus-type algorithms for networked systems whose topologies switch randomly. The regime-switching process is modeled as a discrete-time Markov chain with a finite state space. The consensus control is achieved by using stochastic approximation methods. In the setup, the regime-switching process (the Markov chain) contains a rate parameter $\varepsilon > 0$ in the transition probability matrix that characterizes how frequently the topology switches. On the other hand, the consensus control algorithm uses a stepsize μ that defines how fast the network states are updated. Depending on their relative values, three distinct scenarios emerge. Under suitable conditions, we show that when $0 < \varepsilon = \mathcal{O}(\mu)$, a continuous-time interpolation of the iterates converges weakly to a system of randomly switching ordinary differential equations modulated by a continuous-time Markov chain. In this case a scaled sequence of tracking errors converges to a system of switching diffusion. When $0 < \varepsilon \ll \mu$, the network topology is almost non-switching during consensus control transient intervals, and hence the limit dynamic system is simply an autonomous differential equation. When $\mu \ll \varepsilon$, the Markov chain acts as a fast varying noise, and only its averaged network matrices are relevant, resulting in a limit differential equation that is an average with respect to the stationary measure of the Markov chain. Simulation results are presented to demonstrate these findings.

1. Introduction

This paper studies convergence properties of consensus-type algorithms for networked systems in which the network topologies switch randomly. Consensus problems are related to many control applications that involve coordination of multiple entities with only limited neighborhood information to reach a global goal for the entire team. Typical examples include multi-agents in robotics, flocking behavior in people and animals, wireless communication networks, sensor networks, platoon formation in ground and aerial vehicles, distributed computing, biological systems, etc. Due to the diversity in application domains, detailed system descriptions vary substantially and diversified methodologies are needed to treat such systems. However, one common feature of the underlying problems is: Although the goal of control is global to the entire system, only limited local information is available for control actions.

There is an extensive literature on consensus control in a variety of application areas, including computing load balancing (Lynch, 1997; Xiao, Boyd, & Kim, 2007), sensor networks (Akyildiz, Su, Sankarasubramniam, & Cayirci, 2002; Ogren, Fiorelli, & Leonard, 2005), mobile agents (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004), flocking behavior and swarms (Liu, Passino, & Polycarpou, 2003; Toner & Tu, 1998; Viseck, Czirook, Ben-Jacob, Cohen, & Shochet, 1995), etc. Related algorithms and theoretical developments were reported in Cortes and Bullo (2005), Huang and Manton (2009) and Ren and Beard (2005). Much recent work was motivated by Viseck et al. (1995), which in fact is a version of a model introduced earlier in Reynolds (1987) for simulating flocking and schooling behaviors in computer graphics. The effort in the control community can be traced back to the asynchronous stochastic optimization algorithms (Tsitsiklis, Bertsekas, & Athans, 1986), which was substantially generalized in Kushner and Yin (1987). In this paper, we consider a specific control structure for average consensus. It is noted that consensus control often leads to consensus without further constraints on the actual state. Practical systems always require states to be confined in some ways. Our link-based control provides a natural and practical constraint on the state. Consider for instance the problem of

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networked computing (Xiao et al., 2007; Yin, Xu, & Wang, 2003). A computational job is assigned to a network of r computers. The goal is to achieve approximately equal workload distribution for each computer to avoid idle or overloaded running states. A workload transfer from node i to node j results in a decrease of workload at node i and an increase of the same amount at node j. This control structure does not change the total workload amount of the whole system and provides a natural constraint to bound the node states. This scenario can be easily recognized in different application domains such as material distribution systems, data fusion in distributed sensor networks, deployment of sensors, coordination of unmanned aerial vehicles (UAVs). It will be shown that this constraint leads to a Markovian dynamic system that connects seamlessly with the Markov chain descriptions of the network topology switching dynamics.

To model inherent uncertainties and time-varying nature in communication networks, we consider consensus control problems with regime-switching network topologies. In our setup, we quantify the time-varying parameter process as a Markov chain with a transition matrix that includes a small parameter $\varepsilon > 0$, which characterizes the rates of network switching. We then use a stochastic recursive algorithm to carry out the consensus control task. The algorithm uses a small stepsize $\mu > 0$, which defines how fast the network node states are updated. The impact of network switching rates on convergence properties of consensus control algorithms is captured by the relationship between ε and μ. There are three cases concerning the relative sizes of ε and μ: $0 < \varepsilon = \mathcal{O}(\mu)$, $0 < \varepsilon \ll \mu$, and $0 < \mu \ll \varepsilon$. Asymptotic behaviors of consensus control algorithms under these cases are fundamentally different. When $\varepsilon = \mathcal{O}(\mu)$, through appropriate interpolations, the limit is described by regime-switching ordinary differential equations. When $\varepsilon \ll \mu$, the network topology rarely changes and is essentially fixed during the transient interval of active consensus control. We thus practically deal with a fixed network. When $\mu \ll \varepsilon$, the network is changing so fast that it acts like a noise, and consequently only its average with respect to the stationary measure determines convergence properties of the consensus control.

Switching network topologies were studied in Moreau (2005), Olfati-Saber, Fax, and Murray (2007), and more recently in Huang, Dey, Nair, and Manton (2010) and Kar and Moura (2009). This paper differs from the existing literature in several essential aspects. Moreau (2005) and Olfati-Saber et al. (2007) do not use Markov formulations. In Huang et al. (2010), the authors considered stochastic consensus over lossy wireless networks, in which the proposed measurement model has a random link gain, an additive noise, and a Markov lossy signal reception; arbitrary switching was also considered there. Kar and Moura (2009) employs randomly switching Laplacian matrices together with observation noises that may be state dependent and Markovian. The Laplacian matrices share a common average. Its main approach is based on convergence of products of stochastic matrices. Thus, system analysis and consensus are established from the averaged network. We treat a more general Markov model and treat a much larger class of noises. In this paper the graph is modulated by a discrete-time Markov chain. In addition to the traditional additive structure of the noise, we allow the noise to be nonadditive, correlated and non-Markovian. The function involved in the nonadditive noise can be time varying and depend on both the analog states and Markov chain states; see the remark section at the end of this paper. In lieu of examining the product of random matrices, our analysis is based on stochastic analysis of random processes. Thus far reaching results are obtained that better delineate the system dynamics and evolution. We establish convergence and rates of convergence of the algorithm, and study the intrinsic properties of the random dynamic systems involved. Interacting with consensus control strategies, we show that the limit system depends on relative speeds of the control and topology switching frequencies, and it may still be a stochastic system whose convergence is much harder to derive. By treating different rates of variation of the control and time-varying Markov parameter, our results depart from typical consensus control conclusions, initiate a multi-scale modeling and analysis, and potentially better reflect the needs of adjusting consensus control strategies in light of topology switching. Furthermore, the expanded classes of noises can cover many communication schemes.

The rest of the paper is organized as follows. First, networked systems and consensus control problems are introduced in Section 2. Some basic properties of networked systems are derived for time-invariant systems, which are to be used in subsequent convergence analysis. Section 3 sets the stage for networked systems with randomly time-varying topologies. The problem formulation of regime-switching network topologies is introduced. Treating such systems are our primary concerns in this paper. Convergence analysis under the scenario $\varepsilon = \mathcal{O}(\mu)$ is presented in Section 4. It is shown that in this case, convergence of the consensus control is governed by a regime-switching ordinary differential equation. Convergence rates are derived by using a centered and scaled sequence of the iterates. Using the weak convergence methods, we prove that a suitably scaled sequence of network states converge to the solution of a regime-switching stochastic differential equation. We also establish certain stability results. Section 5 focuses on convergence analysis for the cases of fast-switching and slow-switching network topologies. These two sections delineate a complete picture of convergence properties of consensus control algorithms. Section 6 provides simulation examples to illustrate the asymptotes. Finally Section 7 concludes the paper with further remarks.

2. Networked systems and consensus control

Consider a networked system of r nodes, given by

$$x^i_{n+1} = x^i_n + u^i_n, \quad i = 1, \ldots, r, \tag{1}$$

where u^i_n is the node control for the ith node, or in a vector form $x_{n+1} = x_n + u_n$ with $x_n = [x^1_n, \ldots, x^r_n]'$, $u_n = [u^1_n, \ldots, u^r_n]'$. The nodes are linked by a sensing network, represented by a directed graph $\mathcal{G}$ whose element (i, j) indicates estimation of the state x^j_n by node i via a communication link, and a permitted control v^{ij} on the link. For node i, $(i, j) \in \mathcal{G}$ is a departing edge and $(l, i) \in \mathcal{G}$ is an entering edge. The total number of communication links in $\mathcal{G}$ is l_s. From its physical meaning, node i can always observe its own state, which will not be considered as a link in $\mathcal{G}$.

2.1. Networked observation and control

In this paper, we limit the control structures to the link control among nodes permitted by $\mathcal{G}$. The node control u^i_n is determined by the link control v^{ij}_n. Since a positive transportation of quantity v^{ij}_n on (i, j) means a loss of v^{ij}_n at node i and a gain of v^{ij}_n at node j, the node control at node i is $u^i_n = -\sum_{(i,j)\in\mathcal{G}} v^{ij}_n + \sum_{(j,i)\in\mathcal{G}} v^{ji}_n$. The most relevant implication in this control scheme is that for all n, $\sum_{i=1}^r x^i_n = \sum_{i=1}^r x^i_0 := \eta r$, for some $\eta \in \mathbb{R}$ that is the average of x_0. That is, $\eta = \sum_{i=1}^r x^i_0 / r$. Consensus control seeks control algorithms that achieve $x_n \to \eta\mathbb{1}$, where $\mathbb{1}$ is the column vector of all 1s. A link $(i, j) \in \mathcal{G}$ entails an estimate, denoted by $\widehat{x}^{ij}_n$, of x^j_n by node i with estimation error d^{ij}_n, i.e.,

$$\widehat{x}^{ij}_n = x^j_n + d^{ij}_n. \tag{2}$$