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Brief paper

Necessary conditions for Hadamard factorizations of Hurwitz polynomials[☆]

Carlos Arturo Loredo-Villalobos, Baltazar Aguirre-Hernández [*]

Departamento de Matemáticas, Universidad Autónoma Metropolitana–Iztapalapa, Av. San Rafael Atlixco no. 186, Col. Vicentina, Iztapalapa 09340, México, D.F., Mexico

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ABSTRACT

A property of Hurwitz polynomials is related with the Hadamard product. Garloff and Wagner proved that Hadamard products of Hurwitz polynomials are Hurwitz polynomials, and Garloff and Shrinivasan shown that there are Hurwitz polynomials of degree 4 which do not have a Hadamard factorization into two Hurwitz polynomials of the same degree 4. In this paper, we give necessary conditions for an even-degree Hurwitz polynomial to have a Hadamard factorization into two even-degree Hurwitz polynomials; such conditions are given in terms of the coefficients of the given polynomial alone. Furthermore, we show that if an odd-degree Hurwitz polynomial has a Hadamard factorization then a system of nonlinear inequalities has at least one solution.

1. Introduction

A lot of information about Hurwitz polynomials has been generated due to their importance for checking the stability of continuous linear systems. Among the criteria for verifying if a given polynomial is a Hurwitz polynomial it is worth mentioning the Routh–Hurwitz criterion (Hurwitz, 1895), although the Hermite–Biehler theorem (Hermite, 1856) and the stability test (Bhattacharyya, Chapellat, & Keel, 1995) are also frequently used. Other criteria and interesting questions about Hurwitz polynomials can be found in Bhattacharyya et al. (1995), Gantmacher (1959) and Lancaster and Tismenetsky (1985). On the other hand, the relationship between the Hadamard product and Hurwitz polynomials has also been studied. Given two polynomials $p(t)$ and $q(t)$ with real coefficients

$$p(t) = \beta_n t^n + \beta_{n-1} t^{n-1} + \cdots + \beta_1 t + \beta_0 \tag{1}$$

$$q(t) = \gamma_n t^n + \gamma_{n-1} t^{n-1} + \cdots + \gamma_1 t + \gamma_0, \tag{2}$$

the Hadamard product of p and q, $p * q$, is defined as follows:

$$(p * q)(t) = \beta_n \gamma_n t^n + \beta_{n-1}\gamma_{n-1} t^{n-1} + \cdots + \beta_1\gamma_1 t + \beta_0\gamma_0.$$

In Garloff and Wagner (1996), it was shown that the set of Hurwitz polynomials is closed under the Hadamard product. On the other hand, in Garloff and Shrinivasan (1996), it was shown that there are Hurwitz polynomials of degree 4 that do not have a Hadamard factorization into two Hurwitz polynomials. Other related questions can be found in Garloff and Wagner (1994). In this paper, we give necessary conditions for a Hurwitz polynomial $f(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_1 t + \alpha_0$ to have a Hadamard factorization $f = g * h$.

2. Main results

In this section, we present necessary conditions for an even-degree Hurwitz polynomial to admit a Hadamard factorization. It is easy to see that all Hurwitz polynomials of degree 2 of 3 have Hadamard factorizations into two Hurwitz polynomials, and besides, it is known that there are Hurwitz polynomials of degree 4 which do not have a Hadamard factorization. In this paper, we present a result which is valid for all even-degree Hurwitz polynomials with positive coefficients. Later, we use this result to present necessary conditions for Hadamard factorizations of odd-degree Hurwitz polynomials.

Theorem 1. *Let $f(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ be an even-degree Hurwitz polynomial with positive coefficients. If $f(t)$ has a Hadamard factorization into two Hurwitz polynomials with positive coefficients, then*

$$\frac{\alpha_{i+1}\alpha_0}{\alpha_1} < \frac{\alpha_{i-1}\alpha_n}{\alpha_{n-1}} + \alpha_i - 2\sqrt{\frac{\alpha_{i-1}\alpha_i\alpha_n}{\alpha_{n-1}}}, \tag{3}$$

$i = 2, 4, \ldots, n-2$.

The following result is immediate.

Corollary 1. *Consider an even-degree Hurwiz polynomial with positive coefficients*

$$f(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_2 t^2 + \alpha_1 t + \alpha_0$$

such that the system of inequalities (3) is not satisfied. Then $f(t)$ does not have a Hadamard factorization.

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[*] Corresponding author. Tel.: +52 58044654; fax: +52 58044660.
E-mail addresses: r2ro.loredo@gmail.com (C.A. Loredo-Villalobos), bahe@xanum.uam.mx (B. Aguirre-Hernández).

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Example 1. Consider the Hurwitz polynomial $f(t) = t^4 + 2t^3 + 3t^2 + 5t + 1$. Here, $\alpha_0 = 1, \alpha_1 = 5, \alpha_2 = 3, \alpha_3 = 2, \alpha_4 = 1$. Then, inequality (3) becomes

$$\frac{2}{5} < \frac{5}{2} + 3 - 2\sqrt{\frac{15}{2}},$$

which is not satisfied. Consequently, $f(t)$ does not have a Hadamard factorization.

Example 2. Consider the Hurwitz polynomial

$$f(t) = t^6 + 36t^5 + 225t^4 + 400t^3 + 225t^2 + 36t + 18.$$

Here, $\alpha_0 = 18, \alpha_1 = \alpha_5 = 36, \alpha_2 = \alpha_4 = 225, \alpha_3 = 400$, and $\alpha_6 = 1$. Then, the system of inequalities (3) becomes

$$\frac{400(18)}{36} < \frac{(36)(1)}{36} + 225 - 2\sqrt{\frac{36(1)(225)}{36}}$$
$$\frac{36(18)}{36} < \frac{400(1)}{36} + 225 - 2\sqrt{\frac{400(1)(225)}{36}}.$$

The first inequality is not satisfied since it is not possible that $200 < 196$. Consequently, $f(t)$ does not have a Hadamard factorization into two Hurwitz polynomials of degree 6.

Example 3. Consider the Hurwitz polynomial

$$f(t) = t^4 + 8t^3 + 9t^2 + 2t + 1.$$

Here, $\alpha_0 = 1, \alpha_1 = 2, \alpha_2 = 9, \alpha_3 = 8$, and $\alpha_4 = 1$. Then, inequality (3) becomes

$$\frac{8}{2} < \frac{2}{8} + 9 - 2\sqrt{\frac{9}{4}},$$

which is satisfied. Then f could have a Hadamard factorization. In fact, f has a Hadamard factorization $f = g * h$ with

$$g(t) = t^4 + 4t^3 + 6t^2 + 4t + 4$$

and

$$h(t) = t^4 + 2t^3 + \frac{3}{2}t^2 + \frac{1}{2}t + \frac{1}{4}.$$

There is a problem in obtaining a similar result for odd-degree Hurwitz polynomials (see the Remark after the proof of Theorem 1). However, in the next theorem, we give necessary conditions for odd-degree polynomials in terms of a system of nonlinear inequalities.

Theorem 2. *Consider $f(t) = \alpha_{2m+1}t^{2m+1} + \alpha_{2m}t^{2m} + \cdots + \alpha_1 t + \alpha_0$, an odd-degree Hurwitz polynomial with positive coefficients. If $f(t)$ has a Hadamard factorization into two Hurwitz polynomials with positive coefficients, then the following system of inequalities has at least one solution $(x_1, \ldots, x_{m-1})$, where $x_k > 0$, $k = 1, \ldots, m-1$:*

$$\begin{aligned}
&\sqrt{\alpha_0}x_1x_{m-1} + \sqrt{\alpha_{2m}} < \sqrt{\alpha_2}x_{m-1}\\
&\sqrt{\alpha_0}x_2x_{m-1} + \sqrt{\alpha_{2m}}x_1 < \sqrt{\alpha_4}x_{m-1}\\
&\sqrt{\alpha_0}x_3x_{m-1} + \sqrt{\alpha_{2m}}x_2 < \sqrt{\alpha_6}x_{m-1}\\
&\vdots\\
&\sqrt{\alpha_0}x_{m-1}^2 + \sqrt{\alpha_{2m}}x_{m-2} < \sqrt{\alpha_{2m-2}}x_{m-1}.
\end{aligned} \tag{4}$$

Example 4. Consider the Hurwitz polynomial

$$f(t) = t^5 + 4t^4 + 3t^3 + 5t^2 + t + 1.$$

Here, $\alpha_5 = 1, \alpha_4 = 4, \alpha_3 = 3, \alpha_2 = 5, \alpha_1 = 1, \alpha_0 = 1$, and $m = 2$. We analyse the inequality $x_1^2 + \sqrt{4} < \sqrt{5}x_1$, that is, $x_1^2 - \sqrt{5}x_1 + 2 < 0$. Since the corresponding discriminant is $(\sqrt{5})^2 - 4(1)(2) = 5 - 8 = -3$, the inequality $x_1^2 - \sqrt{5}x_1 + 2 < 0$ is not satisfied for $x_1 > 0$, and the polynomial $f(t)$ does not have a Hadamard factorization.

Remark 1. In fact, the existence of one solution of (4) is equivalent to having $0 < \alpha_{2m-2}\alpha_2 - \alpha_{2m}\alpha_0$. Suppose that (4) is satisfied. Then, from the first and last inequalities, we have that

$$\sqrt{\alpha_{2m}} < \sqrt{\alpha_0}x_1x_{m-1} + \sqrt{\alpha_{2m}} < \sqrt{\alpha_2}x_{m-1}$$

and

$$\sqrt{\alpha_0}x_{m-1}^2 < \sqrt{\alpha_0}x_{m-1}^2 + \sqrt{\alpha_{2m}}x_{m-2} < \sqrt{\alpha_{2m-2}}x_{m-1};$$

that is,

$$\sqrt{\alpha_{2m}} < \sqrt{\alpha_2}x_{m-1} \quad \text{and} \quad \sqrt{\alpha_0}x_{m-1}^2 < \sqrt{\alpha_{2m-2}}x_{m-1}.$$

Thus,

$$\frac{\sqrt{\alpha_{2m}}}{\sqrt{\alpha_2}} < x_{m-1} < \frac{\sqrt{\alpha_{2m-2}}}{\sqrt{\alpha_0}}, \tag{5}$$

and consequently $0 < \alpha_{2m-2}\alpha_2 - \alpha_{2m}\alpha_0$. On the other hand, if $0 < \alpha_{2m-2}\alpha_2 - \alpha_{2m}\alpha_0$, then

$$\frac{\sqrt{\alpha_{2m}}}{\sqrt{\alpha_2}} < \frac{\sqrt{\alpha_{2m-2}}}{\sqrt{\alpha_0}}.$$

Clearly, we can find $x_{m-1} > 0$ such that (5) holds. It can be seen that x_{m-1} satisfies $\sqrt{\alpha_{2m}} < \sqrt{\alpha_2}x_{m-1}$ and $\sqrt{\alpha_0}x_{m-1}^2 < \sqrt{\alpha_{2m-2}}x_{m-1}$; then we can find $x_1, x_2, \ldots, x_{m-2} > 0$, which are small enough such that (4) is satisfied. Then we have the next result.

Corollary 2. *If the odd-degree Hurwitz polynomial with positive coefficients $f(t) = \alpha_{2m+1}t^{2m+1} + \alpha_{2m}t^{2m} + \cdots + a_1t + \alpha_0$ has a Hadamard factorization, then*

$$0 < \alpha_{2m-2}\alpha_2 - \alpha_{2m}\alpha_0 \tag{6}$$

is satisfied.

3. Applications

In this section, we show how our results are useful in studying some problems in control theory and its applications.

3.1. The crane

On p. 3 in Ackerman (2002), it is shown that the characteristic polynomial of a crane is a polynomial of degree 4. On p. 337, for some values of the parameters, the following polynomial is obtained: $f(t, k_3) = 12\,000t^4 + 34\,200t^3 + (31\,000 - k_3)t^2 + 28\,500t + 5000$. For $k_3 = 16\,000$, this becomes the Hurwitz polynomial $f(t, 16\,000) = 12\,000t^4 + 34\,200t^3 + 15\,000t^2 + 28\,500t + 5000$. Here, $\alpha_0 = 5000, \alpha_1 = 28\,500, \alpha_2 = 15\,000, \alpha_3 = 34\,200$, and $\alpha_4 = 12\,000$. Then, inequality (3) becomes $6000 < 505.10258$, which is not satisfied. Consequently, $f(t, 16\,000)$ does not have a Hadamard factorization.

Remark 2. It would be very important to discover the physical meaning that the Hadamard factorization has for the polynomial associated with the model of a real problem. This might be the subject of future research.