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Global asymptotic stabilization of the attitude and the angular rates of an underactuated non-symmetric rigid body☆

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Abstract

The paper deals with the global stabilization of both the attitude and the angular velocities of an underactuated rigid body. First a stability theorem is proven for a class of systems; subsequently, the equations describing the physics of the rigid body are presented, showing that the rigid body belongs to the considered class of systems, and a sufficient condition for the application of the theorem to the stability of the rigid body equilibrium is pointed out. Finally, some simulation results are reported showing the effectiveness of the proposed methodology.

Keywords: Rigid body; Switching control; Lyapunov function

1. Introduction

In the past, the problem of asymptotic stabilization of the zero equilibrium of the rigid body attitude has been widely studied. For instance Crouch (1984) provides necessary and sufficient conditions for the system to be controllable and, based on this analysis, devises an algorithm for which the zero equilibrium of the linearized system is locally stabilized. Byrnes and Isidori (1991) have proven that *no continuous stabilizing feedback exists* for the underactuated rigid body and they propose a feedback control law that locally asymptotically drives the rigid body to a motion around one of the axes. Krishnan, Reynahoglu, and McClamroh (1992), similarly to Crouch, iteratively apply available controllability techniques in a sort of closed loop law. Local asymptotic stability has also been achieved by Morin, Samson, Pomet, and Jiang (1995) by using the Centre Manifold Theorem to derive a smooth time-varying control law and, more in general, by Coron and Keraï (1996).

Other works focus on the stabilizability of the velocities; Astolfi (1999), for instance, proves that a suitable "practical" exponential stability property can be also achieved when only some of the states are measurable. As far as the underactuated rigid body is concerned, Aeyels and Szafranski (1988) have shown that a single torque cannot asymptotically stabilize the zero equilibrium, while Aeyels (1985) shows that the simpler problem of practical stabilization is solvable even if only one torque is available. Furthermore, the case of a symmetric rigid body has been considered by Sontag and Sussmann (1988) and by Outbib and Sallet (1992).

In this paper the simultaneous stabilization of both the attitude and the angular velocities is considered; the proposed solution is based on a novel approach, the main feature of which can be described as follows. Consider a more generic version of the equations describing the physics of the rigid body, i.e. a system constituted by two equations, $\dot{\boldsymbol{\omega}} = \mathbf{F}(\boldsymbol{\omega}, \mathbf{w})$ and $\dot{\boldsymbol{\xi}} = \mathbf{G}(\boldsymbol{\omega}, \boldsymbol{\xi})$, which are referred to as *dynamic* and *kinematic* equations, respectively. Suppose that the *dynamic* equation is completely controllable by means of the input $\mathbf{w}$ while the *kinematic* equation is not directly affected by the input and is, instead, affected by the dynamic state variable

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$\boldsymbol{\omega}$. Moreover, suppose that $\boldsymbol{\omega} = \mathbf{0}$ is an equilibrium state for the *dynamic* equation and that $\boldsymbol{\xi}_0$ is an equilibrium state for the *kinematic* equation. If there exists a feedback $\boldsymbol{\omega}^*(\boldsymbol{\xi})$ for the *kinematic* equation such that $\boldsymbol{\omega}^*(\boldsymbol{\xi}(t))$ tends to zero when $t \to \infty$ and such that $\boldsymbol{\omega} = \boldsymbol{\omega}^*$ renders the equilibrium in $\boldsymbol{\xi}_0$ asymptotically stable, then the problem of asymptotic stability of the equilibrium $(\mathbf{0}, \boldsymbol{\xi}_0)$ for the pair of equations is "reduced" to the search for the input $\mathbf{w}^*$ such that $\mathbf{F}(\boldsymbol{\omega}^*, \mathbf{w}^*) = \dot{\boldsymbol{\omega}}^*$. Now, the reduced problem may turn out to be rather simple when $\mathbf{F}(\cdot,\cdot)$ has a simple structure, for instance $\mathbf{F}(\boldsymbol{\omega}, \mathbf{w}) = A\mathbf{w}$, where $A \in \mathbb{R}^{n\times n}$ is an invertible matrix with constant coefficients, as in a backstepping scheme. On the contrary, in other cases finding a solution may be very difficult, as in the case of the underactuated dynamics of a non-symmetric rigid body that will be analyzed in the following, or even impossible.

However, the successful idea on which the main result of the paper rests is considering only piecewise constant stabilizing functions $\boldsymbol{\omega}^*$, which allows us to explicitly find the (sequence of) inputs solving the reduced problem for the underactuated rigid body dynamics.

In a second step of the analysis, the hypothesis of piecewise constant values is released: a sufficient condition is pointed out for a piecewise *almost*-constant (in the sense that will be specified in the following) $\boldsymbol{\omega}^*$ to globally asymptotically stabilize the zero equilibrium of the *kinematic* equation, while maintaining the condition that $\boldsymbol{\omega}^*$ tends to zero, as $\boldsymbol{\xi}$ does.

In the next section the main problem is formulated and in Section 3 a piecewise constant sequence of values of $\boldsymbol{\omega}$ is shown to globally asymptotically stabilize the zero-equilibrium of the *kinematic* equation. In Section 4 the generic class of systems constituted by the pair of a *dynamic* and a *kinematic* equations is considered. In Section 5 some simulation results are reported showing the effectiveness of the proposed switching control law and, finally, in Section 6 some concluding remarks are drawn.

2. Problem formulation

This section is dedicated to present the equations describing the variation of the angular rates (dynamic equations) and of the attitude (kinematic equations) of an underactuated rigid body. The reader is referred to the available literature (see, for example, Goldstein, Poole, and Safko (2002) and Wertz (1978)) for more details.

The acceleration of a rigid body, the motion of which in the three-dimensional space is free, can be expressed as

$$\begin{cases} J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 = M_x, \\ J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 = M_y, \\ J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 = M_z, \end{cases} \tag{1}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^\top$ is the rotation velocity of a coordinates frame fixed to the body (w.r.t. the fixed coordinate reference x–y–z), $\mathbf{M} = (M_x, M_y, M_z)^\top$ is the torque acting on the body and J_1, J_2 and J_3 are the main inertia moments w.r.t. the frame fixed to the body.

Clearly, the system described by (1) is underactuated if one of the terms on the right-hand side is zero; in the following, without loss of generality, the case $M_z = 0$ is considered. Letting $u \triangleq ((J_2 - J_3)\omega_2\omega_3 + M_x)/J_1$, $v \triangleq ((J_3 - J_1)\omega_3\omega_1)/J_2$ and $a = (J_1 - J_2)/J_3$, system (1) becomes

$$\dot{\omega}_1 = u, \qquad \dot{\omega}_2 = v, \qquad \dot{\omega}_3 = a\omega_1\omega_2, \tag{2}$$

where, u and v are the control variables. In Eq. (2) the assumption $a \neq 0$ is necessary in order to avoid the trivial equation $\dot{\omega_3} = 0$. The condition $a \neq 0$, or equivalently $J_x \neq J_y$, implies that a non-symmetric rigid body is considered.

The equations describing the kinematics can be written in several different ways, according to the chosen parameterization (Wertz, 1978). In the following, the unit quaternion is used, namely a vector $\boldsymbol{\xi} \in \mathbb{R}^4$ such that $\|\boldsymbol{\xi}\| = 1$, allowing a global representation of the rigid body attitude. With respect to this kind of parameterization, the equations are (e.g. Lovera and Astolfi (2004))

$$\dot{\boldsymbol{\xi}} = G(\boldsymbol{\omega})\boldsymbol{\xi} \tag{3}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^\top$ and

$$G(\boldsymbol{\omega}) = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}. \tag{4}$$

The problem addressed in the paper is the *global asymptotic stabilization* of the equilibrium $\boldsymbol{\xi}_0 = (0, 0, 0, 1)^\top$, $\boldsymbol{\omega} = \mathbf{0}$, of the system described by Eqs. (2) and (3).

3. A stabilizing control strategy for the kinematic equations

In this section the dynamics of the rigid body is temporarily neglected, and the reasoning is focused only on the *kinematic* equation (3). Moreover, a discontinuous time-behaviour of $\boldsymbol{\omega}$ is supposed to be allowed, what is indeed a purely ideal assumption. More specifically, for $i \in \{1, 2, 3\}$ the following set is defined

$$\mathcal{O}_i \triangleq \left\{\boldsymbol{\omega} \text{ such that } \omega_i \neq 0 \text{ and } \omega_j = 0 \text{ for } j \neq i\right\}$$

and the following temporary assumption is made.

Assumption 3.1. For every pair $i, j \in \{1, 2, 3\}$, there exists a pair of inputs which applied to (2) cause an instantaneous variation of the vector of the angular rates from a value in $\mathcal{O}_j$ to a value in $\mathcal{O}_i$.

In the rest of the section the problem of stabilizing the equilibrium in $\boldsymbol{\xi}_0$ is proven to be solvable by means of a switching control law.

3.1. Specification of the switching control scheme

The generic structure of a switching control scheme can be summarized as follows: a set of feedback control laws is available and at each time-instant one of them is selected, to be used in the closed loop, by a *decision maker* according to a given *switching strategy*.

Thus, a switching control scheme behaves as a finite state automaton to each state of which a control law is assigned.