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Brief paper

Simultaneous estimation of the input and output frequencies of nonlinear systems☆

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Abstract

In this paper, the simultaneous estimation of the input and output frequencies of nonlinear systems is considered. As the output frequencies are generated from the input frequencies, and are integer combinations of these frequencies, it is shown in this paper that the simultaneous estimation of both the input and output frequencies can therefore be formulated as a constrained estimation problem. First, the constrained Cramér–Rao lower bound, an important general property of any unbiased estimator, is derived. The procedure and algorithm for estimating the input and output frequencies are devised based on the periodogram method. Numerical examples are presented to illustrate the performance and implementation of the proposed estimation procedure and algorithm.

Keywords: Frequency estimation; Cramér–Rao lower bound; Periodogram; Simplex search method

1. Introduction

The estimation of the frequencies of sinusoidal functions from a finite number of noisy discrete measurements has attracted great attention. A number of techniques have been developed and are available in the literature, e.g., the maximum likelihood method (Rife & Boorstyn, 1976), the nonlinear least squares (Stoica & Nehorai, 1988), the linear prediction (So, Chan, Chan, & Ho, 2005), the periodogram method (Quinn & Hannan, 2001), MUSIC (Schmidt, 1986) and ESPRIT (Roy & Kailath, 1989).

It is well known that there exists a simple relationship between the input and the output frequencies of a linear system. However, this relationship becomes more complex for nonlinear systems, as it depends on the order of nonlinearity, which for simplicity is defined as the highest order of the polynomial approximation of the nonlinearity of the system. For these nonlinear systems, the output frequencies are an integral combination of the input frequencies, as the output is generated by the input (Lang & Billings, 2000). If the input frequencies and the order of the nonlinearity are known, then the output frequencies can be readily derived. However, if the input frequencies and/or the order of the nonlinearity are unknown, as is often the case in practice, it is necessary to estimate the input and output frequencies from the measurements of the input and output. A common approach is to estimate the frequencies of the input using only the measurements of the input, and that of the output using only the measurements of the output without taking into account the relationship between the input and output frequencies. It is shown in this paper that estimating simultaneously both the input and output frequencies for a nonlinear system excited by multi-tone sinusoidal signals is more accurate than estimating these frequencies separately.

In this paper, the simultaneous estimation of both the input and output frequencies is formulated as a constrained estimation problem. A three-step procedure is proposed to solve this estimation problem. First, initial estimates of the input and output frequencies are obtained by the periodogram method (Quinn & Hannan, 2001), and from which the order of the nonlinearity of the system is estimated. The final estimates of the input and output frequencies are obtained by maximizing

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an objective function using the simplex search method. As the Cramér–Rao lower bound (*CRLB*) is the lower bound of the variance of any unbiased estimator, it is used here as a benchmark for comparing the performance of the proposed method and methods that estimate these frequencies separately. It is shown in this paper that the *CRLB* for the proposed method, referred simply as the constrained *CRLB* in the following analysis, is smaller than that estimating these frequencies separately, indicating better estimates are obtained using the proposed method.

This paper is organized as follows. In Section 2, the preliminary assumptions and results for the estimation of the input and output frequencies are presented. This is followed by a formal definition of the problem in Section 3. The *CRLB* for both separate and simultaneous estimation of the input and output frequencies are given in Section 4. The proposed three-step frequency estimation procedure is described in Section 5. A numerical example is presented in Section 6 to illustrate the performance of the proposed method.

2. Preliminaries

Consider an asymptotically stable continuous single-input single-output nonlinear system given by the state space equation (Chua & Ushida, 1981),

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \bar{u}) \\ \bar{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{1}$$

where $\mathbf{x}$ is the state vector with appropriate dimension, $\bar{y}$ is the output, and $\bar{u}$ is the R-tone sinusoidal input. Let $\bar{u}$ be given by,

$$\bar{u}(t) = \boldsymbol{\alpha}^{\mathrm{T}} \sin(\boldsymbol{\omega}_c t + \boldsymbol{\varphi}) \tag{2}$$

where, $\boldsymbol{\alpha} \in R^{R\times 1}$, $\boldsymbol{\omega}_c \in R_+^{R\times 1}$ and $\boldsymbol{\varphi} \in (0,\pi)^{R\times 1}$ are respectively the amplitude, frequency and phase angle of the input. The following assumption is made, which is a slight relaxation of the asymptotically almost periodic assumption proposed in Chua and Ushida (1981).

Assumption 1. For a given initial state $\mathbf{x}_0$, system (1) has a unique asymptotically almost periodic solution,

$$\mathbf{x}(t) = \mathbf{x}_{ss}(t), \quad \text{as } t \to \infty. \tag{3}$$

It follows from Assumption 1 that the steady-state output of (1) is,

$$\bar{y}(t) = \mathbf{C}\mathbf{x}_{ss}(t) = \beta_0 + \boldsymbol{\beta}^{\mathrm{T}} \sin(\boldsymbol{\lambda}_c t + \boldsymbol{\psi}) \tag{4}$$

where β_0 is a constant, $\boldsymbol{\beta} \in R^{S\times 1}$, $\boldsymbol{\lambda}_c \in R_+^{S\times 1}$ and $\boldsymbol{\psi} \in (0,\pi)^{S\times 1}$ are respectively the amplitude, frequency and phase angle of the output. Since the system is assumed to be asymptotically stable, the steady-state output consists of S-tone sinusoids completely generated by the R-tone sinusoidal input. Although S is usually much larger than R, and can even be infinity, it is often set to some arbitrarily large value in practice.

If the input and output frequencies $\boldsymbol{\omega}_c$ and $\boldsymbol{\lambda}_c$ are unknown, they are estimated from noisy discrete measurements of the input and output obtained at a sampling interval of τ. Denote the noisy input and output by $u(t)$ and $y(t)$ respectively,

$$\begin{aligned} u(n) &= \bar{u}(n\tau) + \varepsilon_u(n\tau) \\ y(n) &= \bar{y}(n\tau) + \varepsilon_y(n\tau) \end{aligned} \qquad n = 1, \ldots, N \tag{5}$$

where $\varepsilon_u(t) \sim N(0, \sigma_u^2)$ and $\varepsilon_y(t) \sim N(0, \sigma_y^2)$ are Gaussian white noises. As $\boldsymbol{\omega}_c$ and $\boldsymbol{\lambda}_c$ can vary from 0 to infinity, it is more convenient to transform the estimation problem as follows. Let $\boldsymbol{\omega} = \tau\boldsymbol{\omega}_c$ and $\boldsymbol{\lambda} = \tau\boldsymbol{\lambda}_c$. Further, if τ is chosen satisfying the sampling theorem (Chen, 2004), then $\boldsymbol{\omega} \in (0,\pi)^{R\times 1}$ and $\boldsymbol{\lambda} \in (0,\pi)^{S\times 1}$. As τ is known, the problem of estimating $\boldsymbol{\omega}_c$ and $\boldsymbol{\lambda}_c$ is now transformed to one that estimates $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$, both of which are within the range of 0 and π.

It is common to estimate $\boldsymbol{\omega}$ from u, and $\boldsymbol{\lambda}$ from y (So et al., 2005; Stoica, Moses, Friedlander, & Soderstrom, 1989). As shown in this paper, the estimate of $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$ can be greatly improved if both u and y are used by taking into account the relationship between the input and output frequencies. It is well known that the sth output frequency λ_s is an integer combination of the input frequency $\boldsymbol{\omega}$ (Chua & Ushida, 1981), as follows,

$$\lambda_s = \left| \sum_{r=1}^{R} m_{sr}\omega_r \right| \tag{6}$$

where $|\cdot|$ denotes the absolute value, and m_{sr} is an integer given by,

$$m_{sr} = v_r - v_{-r} \tag{7}$$

and v_r and v_{-r} are respectively the rth and $-r$th element of the "frequency-mix vector", $\mathbf{v}$ (Lang & Billings, 2000; Yue, Billings, & Lang, 2005):

$$\mathbf{v} = [v_{-R}, \ldots, v_{-1}, v_1, \ldots, v_R]. \tag{8}$$

Given a nonlinear system up to $p_{\max}$ order of nonlinearity, for the pth-order nonlinearity, there is a set of $\mathbf{v}$ satisfying the following equality, where the order of the nonlinearity is regarded as the order of the polynomial approximation of the input and output relationship of the nonlinear system at steady state,

$$\{\mathbf{v} : v_{-R} + \cdots + v_{-1} + v_1 + \cdots + v_R = p\}. \tag{9}$$

Rewrite (6) into matrix form,

$$\boldsymbol{\lambda} = \mathbf{M}\boldsymbol{\omega} \tag{10}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_S]^{\mathrm{T}}$, $\boldsymbol{\omega} = [\omega_1, \omega_2, \ldots, \omega_R]^{\mathrm{T}}$, $\mathbf{m}_i = [m_{i1}, m_{i2}, \ldots, m_{iR}]$ is a row vector of $\mathbf{M}$, sign (.,.) is the sign of the product of the arguments, and $\mathbf{M}$, the "Combination Matrix":

$$\mathbf{M} = \begin{bmatrix} \operatorname{sign}(\mathbf{m}_1, \boldsymbol{\omega})\mathbf{m}_1 \\ \operatorname{sign}(\mathbf{m}_2, \boldsymbol{\omega})\mathbf{m}_2 \\ \vdots \\ \operatorname{sign}(\mathbf{m}_s, \boldsymbol{\omega})\mathbf{m}_s \end{bmatrix} \tag{11}$$

where $\mathbf{M} \in Z^{S\times R}$ is an integer matrix. An example is now presented to illustrate how $\mathbf{M}$ is constructed.