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Brief paper

A duality-based convex optimization approach to L_2-gain control of piecewise affine slab differential inclusions[☆]

Behzad Samadi[a], Luis Rodrigues[b,*]

[a] Department of Electrical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

[b] Department of Mechanical and Industrial Engineering, Concordia University, Montreal, QC, H3G 1M8, Canada

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ABSTRACT

This paper introduces the dual parameter set of a piecewise affine (PWA) system. This is a key concept to enable a convex formulation of PWA controller synthesis for PWA slab differential inclusions using a new convex relaxation. Another important contribution of the paper is to present PWA L_2-gain analysis and synthesis results for PWA systems whose output is also a PWA function of the state (as opposed to a piecewise-linear (PWL) function). Unlike other results existing in the literature, the sufficient LMI conditions in this paper are valid for synthesis, even when the PWA systems include sliding modes. A numerical example with sliding modes illustrates the new approach.

1. Introduction

PWA models can describe a large class of nonlinear systems, including linear systems with certain memoryless nonlinearities, such as saturation and dead-zone. In addition to being a broad and complex class of models, PWA systems are locally linear, which enables one to formulate *sufficient* conditions for the stability and performance analysis of PWA systems as convex optimization problems subject to linear matrix inequalities (LMI) (Boyd, Ghaoui, Feron & Balakrishnan, 1994). This has been the trend of research in the work on stability analysis of PWA systems based on Lyapunov functions and LMIs (Decarlo, Branicky, Pettersson & Lennartson, 2000; Gonçalves, Megretski & Dahleh, 2003; Hassibi & Boyd, 1998; Johansson & Rantzer, 1998; Rodrigues, 2004). The analysis of L_2-gain performance of PWA systems is also formulated as a set of LMIs in Hassibi and Boyd (1998), Johansson (2003) and Rantzer and Johansson (2000).

In addition to the work on analysis, controller synthesis for L_2-gain performance of PWA systems has also attracted growing attention (Feng, 2002, 2004; Feng, Lu & Zhou, 2002) and, more recently, controller synthesis based on input to state stability has also been considered in Heemels, Weiland, Juloski and Bemporard (2007). In his work on L_2-gain controller synthesis for uncertain PWA systems, Feng (2002) formulates the synthesis problem as a set of LMIs based on a piecewise quadratic (PWQ) Lyapunov function provided that the structure of the PWA controller was constrained. Feng et al. (2002) proposed a method to design PWL controllers for PWL systems based on a PWQ Lyapunov function to limit the L_2-gain of the system. The method was later extended to uncertain PWL systems in Feng (2004). However, the approaches in Feng (2004) and Feng et al. (2002) do not use any S-procedure in the design process, which means that each closed-loop subsystem of the PWL system has to be stable and this makes the proposed methods conservative. They also ignore attractive sliding modes. There is therefore no guarantee for the closed-loop system to be stable in general.

A very important subclass of PWA systems is the class of PWA *slab systems* (Rodrigues & Boyd, 2005), for which the partition of the state space is a function of a scalar variable. The synthesis of PWL controllers for stability and performance of PWA slab systems is formulated in Hassibi and Boyd (1998) as a set of LMIs. However, for PWA controllers, it is said in Hassibi and Boyd (1998) that "*It doesn't seem that the condition for stabilizability using this type of input command can be cast as an LMI*". Rodrigues and Boyd (2005) showed that by considering an affine term in the controller, the synthesis problem for PWA slab systems can be formulated as a set of LMIs parametrized by a vector. Three different algorithms for controller synthesis have been proposed in Rodrigues and Boyd (2005) and the bisection method has been used to find the controller that maximizes the decay rate of the trajectories.

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* Corresponding author. Tel.: +1 514 8482424x3135; fax: +1 514 8483175.
E-mail addresses: bsamadi@aut.ac.ir (B. Samadi), luisrod@encs.concordia.ca (L. Rodrigues).

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To the best of our knowledge, no convex optimization problem has been proposed for PWA controller design for stability and performance without limiting the structure of the controller. To fill this gap in the literature, this paper formulates PWA controller synthesis for PWA slab systems as a set of LMIs. This result is based on a new key concept: the *dual parameter set* for PWA slab differential inclusions that is introduced in this paper. Considering PWA slab differential inclusions (as opposed to equations) enables the design for stability and performance of nonlinear systems that can be *included* by a PWA envelope.

The structure of the paper is as follows. Some mathematical preliminaries and PWA slab differential inclusions are introduced in Sections 2 and 3, respectively. Performance analysis is presented in Section 4. Section 5 addresses L_2-gain control design. Finally, a numerical example is shown in Section 6 and conclusions are drawn in Section 7.

2. Mathematical preliminaries

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ y = h(x) \end{cases} \tag{1}$$

where $f(x)$ and $g(x)$ are defined *almost everywhere* and bounded for bounded $\|x\|$.

The L_2 gain from w to y is defined as

$$\sup_{0<\|w\|_2<\infty} \frac{\|y\|_2}{\|w\|_2} \tag{2}$$

where the L_2 norm of a signal z is defined as

$$\|z\|_2 = \left[\int_0^\infty z^{\mathrm{T}}(\tau)z(\tau)\mathrm{d}\tau\right]^{\frac{1}{2}} \tag{3}$$

and the supremum is taken over all nonzero trajectories assuming $x_0 = 0$.

The notion of storage function originates in the early work on dissipativity theory started by Willems (1972). The following sufficient condition for finite L_2 gain uses this notion.

Definition 1. The nonlinear system (1) has finite L_2-gain less than $\gamma > 0$ if there exists a locally bounded function $V : \mathbb{R}^n \to \mathbb{R}$, called *storage function*, such that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, $V(0) = 0$ and

$$\forall t \geq 0, \quad V(x(t)) \leq V(x_0) + \int_0^t W(\tau)\mathrm{d}\tau \tag{4}$$

where $W(\tau) = -\|y(\tau)\|^2 + \gamma^2\|w(\tau)\|^2$ is called the *supply rate*.

3. Polytopic PWA slab differential inclusions

Polytopic PWA slab differential inclusions are a generalization of polytopic linear differential inclusions in Boyd et al. (1994). A polytopic PWA slab differential inclusion is described by

$$\begin{aligned} \dot{x}(t) &= A(x,t)x + a(x,t) + B_u(x,t)u + B_w(x,t)w \\ y(t) &= C(x,t)x + c(x,t) + D_u(x,t)u + D_w(x,t)w \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and $y(t) \in \mathbb{R}^{n_y}$ is the output. The initial state is $x(0) = x_0$. It is assumed that system (5) satisfies

$$\begin{aligned} \dot{x} &\in \mathrm{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w, \kappa = 1, 2\} \\ y &\in \mathrm{Conv}\{C_{i\kappa}x + c_{i\kappa} + D_{u_{i\kappa}}u + D_{w_{i\kappa}}w, \kappa = 1, 2\} \end{aligned} \tag{6}$$

for $x \in \mathcal{R}_i$ where Conv stands for the convex hull of a set and $\mathcal{R}_i$, $i = 1, \ldots, N$ are N slab regions partitioning the cross product of a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ and the space of the exogenous input $\mathcal{W}$, defined as

$$\mathcal{R}_i = \{(x, w) \mid \sigma_i < C_{\mathcal{R}}x + D_{\mathcal{R}}w < \sigma_{i+1}\}, \tag{7}$$

where $C_{\mathcal{R}} \in \mathbb{R}^{1\times n}$, $D_{\mathcal{R}} \in \mathbb{R}^{1\times n_w}$ and σ_i for $i = 1, \ldots, N+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{N+1}. \tag{8}$$

It is assumed that $a_{i\kappa} = 0$ and $c_{i\kappa} = 0$ for $i \in \mathcal{I}(0, 0)$ and $\kappa = 1, 2$ where

$$\mathcal{I}(x, w) = \{i \mid (x, w) \in \overline{\mathcal{R}}_i\} \tag{9}$$

and $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Note that if $(x, w) \in \mathcal{R}_i$, then $\mathcal{I}(x, w) = \{i\}$. Each slab region can be described by the following degenerate ellipsoid

$$\mathcal{R}_i = \{(x, w) \mid |L_ix + l_i + M_iw| < 1\} \tag{10}$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$, $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$ and $M_i = 2D_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$.

We consider the following definition of solutions for PWA slab differential inclusions

Definition 2. An absolutely continuous function $x(t)$ is defined to be a solution of (5) if $x(t) \in \mathcal{X}$, $\forall t \geq 0$ and it satisfies

$$\dot{x}(t) \in \mathcal{F}(x, u, w) \tag{11}$$

for *almost all* $t \geq 0$ where

$$\mathcal{F}(x, u, w) \triangleq \mathrm{Conv}\left\{ f_{i\kappa} \,\middle|\, \begin{matrix} f_{i\kappa} = A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w \\ \text{for all } i \in \mathcal{I}(x, w), \kappa = 1, 2 \end{matrix} \right\}. \tag{12}$$

Remark 3. Johansson (2003) defines a PWA differential inclusion as

$$\begin{cases} \dot{x} = A_i(t)x(t) + a_i(t) + B_i(t)u(t) \\ y(t) = C_i(t)x(t) + c_i(t) + D_i(t)u(t), \end{cases} \quad \text{for } x \in \overline{\mathcal{R}}_i \tag{13}$$

where

$$\begin{bmatrix} A_i(t) & a_i(t) & B_i(t) \\ C_i(t) & c_i(t) & D_i(t) \end{bmatrix} = \sum_{\kappa=1}^{\mathcal{K}_i} \alpha_\kappa(t) \begin{bmatrix} A_{i\kappa} & a_{i\kappa} & B_{i\kappa} \\ C_{i\kappa} & c_{i\kappa} & D_{i\kappa} \end{bmatrix} \tag{14}$$

with $\alpha_\kappa(t) \geq 0$ and $\sum_{k=1}^{\mathcal{K}_i} \alpha_\kappa(t) = 1$. A solution for the inclusion (13) is defined in Johansson (2003) as an absolutely continuous function $x(t)$ such that for almost all $t \geq 0$, it satisfies

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in \mathrm{Conv}\left\{ \begin{bmatrix} A_{i\kappa} & a_{i\kappa} & B_{i\kappa} \\ C_{i\kappa} & c_{i\kappa} & D_{i\kappa} \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \\ u(t) \end{bmatrix} \middle|\, k = 1, \ldots, \mathcal{K}_i \right\} \tag{15}$$

for $x(t) \in \overline{\mathcal{R}}_i$.

In this paper, we consider the state equation and the output equation separately as it is seen in (6). Therefore (6) introduces a larger class of inclusions. In particular, the definition of solutions for PWA differential inclusions considered in this paper accommodates attractive sliding modes for which $x(t)$ stays at the boundary of two or more regions for a nonzero time interval.

The next section addresses performance analysis of PWA differential inclusions.

4. Analysis

In this section, performance analysis of PWA differential inclusions is considered. The concept of the *parameter set* of a PWA