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Synchronization of multi-agent systems without connectivity assumptions[☆]

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ABSTRACT

Multi-agent systems arise from diverse fields in natural and artificial systems, such as schooling of fish, flocking of birds, coordination of autonomous agents. In multi-agent systems, a typical and basic situation is the case where each agent has the tendency to behave as other agents do in its neighborhood. Through computer simulations, Vicsek, Czirók, Ben-Jacob, Cohen, and Sochet (1995) showed that such a simple local interaction rule can lead to a certain kind of cooperative phenomenon (synchronization) of the overall system, if the initial states are randomly distributed and the size of the system population is large. Since this model is of fundamental importance in understanding the multi-agent systems, it has attracted much research attention in recent years. In this paper, we will present a comprehensive theoretical analysis for this class of multi-agent systems under a random framework with large population, but without imposing any connectivity assumptions as did in almost all of the previous investigations. To be precise, we will show that for any given and fixed model parameters concerning with the interaction radius r and the agents' moving speed v, the overall system will synchronize as long as the population size n is large enough. Furthermore, to keep the synchronization property as the population size n increases, both r and v can actually be allowed to decrease according to certain scaling rates.

1. Introduction

The collective behavior of multi-agent systems, such as swarm intelligence, consensus, coordination, is a major focus of current research on complex systems, and it has drawn much attention from researchers in diverse fields, including biology (O'Brien, 1989; Okubo, 1986; Parrish, Viscido, Grünbaum, 2002; Shaw, 1975), physics (Vicsek et al., 1995), mathematics (Cucker & Smale, 2007), computer science (Reynolds, 1987), and control theory (Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-Saber, 2006; Ren & Beard, 2005; Savkin, 2004). Scientifically, how locally interacting agents lead to collective behavior of the overall multi-agent systems is a basic and challenging problem to be understood.

Of course, different local rules will give rise to different collective behavior. In this paper, we will study the following basic multi-agent systems: n autonomous agents moving in the plane with the same constant speed and with the heading of each agent updated according to the averaged direction of its neighbors. This model reflects a typical phenomenon in multi-agent systems: each agent has the tendency to behave as other agents do in its neighborhood (O'Brien, 1989; Vicsek et al., 1995). Vicsek et al. (1995) used this model to investigate the gathering, transport and phase transition in nonequilibrium systems, and they also pointed out its potential applications in biological systems involving clustering and migration. Through computer simulations, Vicsek et al. showed that the above system will synchronize when the population density is large and the noise is small. This model looks simple, but the nonlinear relationship in the model makes the theoretical analysis quite hard. In a well-known work, Jadbabaie et al. (2003) initiated a theoretical study for the synchronization of a related model, and inspired much subsequent theoretical investigations on similar problems (see, Cucker and Smale (2007), Liu and Guo (2008a), Moreau (2005), Ren and Beard (2005) and Savkin (2004) among others). What Jadbabaie et al. (2003) showed was that the system will synchronize if the associated dynamical neighbor graphs are jointly connected within some contiguous and bounded time intervals. It is worth mentioning that a similar theoretical result was presented in an earlier paper by Tsitsiklis, Bertsekas, and Athans (1986), but in a rather different context. However, how to remove or verify the troublesome connectivity condition imposed on the underlying dynamical systems turns out to be a difficult and challenging issue in theory, due to the strongly nonlinearly coupled dynamical equations describing the positions and headings of all the agents.

A preliminary step towards the above issue has been made by Liu and Guo (2008a), where a sufficient parameter condition

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is given for the connectivity and hence the synchronization of the system in a deterministic framework with initial headings lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Some counterexamples are also given by Han, Li, and Guo (2006) and Liu and Guo (2008a) to show that the connectivity of the associated neighbor graphs is not sufficient for synchronization if the initial headings are allowed to be in $[0, 2\pi)$. The main problem with Liu and Guo (2008a) is that the condition on the model parameters used there is rather restrictive. There are also a few other papers which establish synchronization without resorting to *a priori* connectivity conditions, by using additional information and/or constrains, for example, Cucker and Smale (2007) studied the model where each agent can interact globally, and Tahbaz-Salehi and Jadbabaie (2007) introduced a periodic boundary condition in the model.

Recently, a major advance towards the complete synchronization analysis is made by Tang and Guo (2007), where a random framework as originally considered by Vicsek et al. (1995) is used in the analysis of the linearized heading equations. They proved that the overall multi-agent system will synchronize with large probability as long as the size of the population is large enough. However, as mentioned by Jadbabaie et al. (2003) and Han et al. (2006), the linearized heading equation may give rise to some unreasonable phenomenon. It is worth pointing out that the random framework considered by Tang and Guo (2007) is just an assumption on the initial distribution of all the agents, the subsequent states together with the associated neighbor graphs, however, may well change from time to time according to the nonlinear dynamical models under consideration. This random framework is obviously different from those studied by Frasca, Carli, Fagnani and Zampieri (2009), Tahbaz-Salehi and Jadbabaie (2008) and Wu (2006), in either the problem formulations or the required assumptions, where certain connectivity conditions are essentially assumed in these papers. However, removing or verifying the troublesome connectivity condition of locally interacting nonlinear multi-agent systems appears to be a "bottleneck" problem in general.

In this paper, we will establish two synchronization theorems for the basic nonlinear model of Vicsek et al. (1995) in the random framework, without changing the locally interacting laws and without imposing any connectivity conditions. In comparison with the linearized heading equations studied by Tang and Guo (2007), a key issue now is how to deal with the difficulties arising from the nonlinear heading equations. We will give a comprehensive theoretical analysis with large population, by using some basic facts of Tang and Guo (2007), together with some estimation for multi-array martingales and with a detailed analysis for the nonlinear equations. Intuitively speaking, large population is beneficial to the connectivity in general, which in conjunction with the averaging mechanism to be given in Eq. (3) will ensure the topology of the dynamical network does not change too much, and hence ensure the synchronization of the system. We will give a rigorous proof for this intuition, and the main results to be established are the following:

(i) For any given and fixed model parameters, *i.e.*, the interaction radius r and the agents' moving velocity v, the overall system will synchronize as long as the population size n is large enough;
(ii) To keep the synchronization property as the population size n increases, both r and v can actually be allowed to decrease according to certain scaling rates to be given in the paper.

Part of the results in this paper was presented in Liu and Guo (2008b) without proof details. The rest of this paper is organized as follows: In Section 2, we will present our main results; Some notations and preliminary lemmas will be given in Section 3; The proofs of the main theorems will be given in Sections 4 and 5 respectively, and some simulation results will be given in Section 6; Finally, some concluding remarks will be made in Section 7.

2. Main results

The multi-agent system to be studied in this paper is composed of n autonomous agents (or subsystems or particles), labeled by $1, 2, \ldots, n$, moving in the plane with the same absolute velocity, and with each agent's heading updated according to the average of the directions of its neighbors (Vicsek et al., 1995). The neighbors of an agent i $(1 \le i \le n)$ at any discrete-time $t = 0, 1, 2, \ldots$ are those which lie within a circle of radius r $(r > 0)$ centered at the agent i's current position. Denote the neighbors of the agent i at time t as $\mathcal{N}_i(t)$, *i.e.*

$$\mathcal{N}_i(t) = \{j | d_{ij}(t) < r\}, \tag{1}$$

where $d_{ij}(t) = \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}$, and $(x_i(t), y_i(t))$ is the position of the agent i at time t. It is easy to see that each agent is a neighbor of itself. Each agent moves in the plane with the same constant absolute velocity v $(v > 0)$, so its position is updated according to the following equation:

$$\begin{cases} x_i(t+1) = x_i(t) + v \cos\theta_i(t+1) \\ y_i(t+1) = y_i(t) + v \sin\theta_i(t+1) \end{cases} \quad \forall i : 1 \le i \le n, \tag{2}$$

where $\theta_i(t)$ is the heading of the agent i at time t, which is updated according to the following average direction of the neighbors' velocity:

$$\theta_i(t+1) = \arctan \frac{\sum_{j \in \mathcal{N}_i(t)} \sin\theta_j(t)}{\sum_{j \in \mathcal{N}_i(t)} \cos\theta_j(t)}, \quad \forall i : 1 \le i \le n. \tag{3}$$

Through computer simulations, Vicsek et al. (1995) showed that the above Eqs. (1)–(3) can make all agents move in the same direction eventually, when the population size n is large enough. Throughout the paper, synchronization means that there exists a constant heading θ, such that

$$\lim_{t\to\infty} \theta_i(t) = \theta, \quad \forall i.$$

From the description of the above mathematical model, we can see that the dynamical behavior of the overall system is determined completely by the moving velocity v, the neighborhood radius r and the initial states. Furthermore, one can observe that the neighbors of each agent are determined by the positions of other agents via (1), whereas the positions of agents are determined by the headings via (2), and moreover, the headings are influenced by the neighbors via (3) in return. So, there is a complicated nonlinear relationship between positions and headings of all agents, which makes a rigorous theoretical analysis quite involved.

The main purpose of this paper is to study the synchronization property of the multi-agent systems (1)–(3) with large population. We will conduct our analysis under the following simple and natural assumptions on the initial states of the system, which are similar to those used in the simulation study of Vicsek et al. (1995).

Assumption 1. The initial positions and headings of all agents are mutually independent, with positions uniformly and independently distributed in the unit square $\mathcal{S}$, and headings uniformly and independently distributed in $[-\pi + \varepsilon_0, \pi - \varepsilon_0]$ with arbitrary $\varepsilon_0 \in (0, \pi)$.

Under Assumption 1, the initial random geometric graph G_0 associated with the initial positions will have some nice properties, one of which is the connectivity studied in the celebrated paper of Gupta and Kumar (1998). Other related nice results may be found in the work of Penrose (2003) and Xue and Kumar (2004). However, in our paper, we need a deeper understanding of both the initial graph and the subsequent dynamic graphs in Sections 3 and 4, which will enable us to establish the following theorem whose proof is given in Section 4.