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The Effective Field Theory approach towards membrane-mediated interactions between particles

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ABSTRACT

Fluid lipid membranes can mediate forces between particles bound to them: A local deformation of the surface geometry created by some object spreads to distant regions, where other objects can respond to it. The physical characteristics of these geometric interactions, and how they are affected by thermal fluctuations, are well described by the simple continuum curvature-elastic Hamiltonian proposed 40 years ago by Wolfgang Helfrich. Unfortunately, while the underlying principles are conceptually straightforward, the corresponding calculations are not—largely because one must enforce boundary conditions for finite-sized objects. This challenge has inspired several heuristic approaches for expressing the problem in a point particle language. While streamlining the calculations of leading order results and enabling predictions for higher order corrections, the ad hoc nature of the reformulation leaves its domain of validity unclear. In contrast, the framework of Effective Field Theory (EFT) provides a *systematic* way to construct a completely equivalent point particle description. In this review we present a detailed account for how this is accomplished. In particular, we use a familiar example from electrostatics as an analogy to motivate the key steps needed to construct an EFT, most notably capturing finite size information in point-like "polarizabilities," and determining their value through a suitable "matching procedure." The interaction (free) energy then emerges as a systematic cumulant expansion, for which powerful diagrammatic techniques exist, which we also briefly revisit. We then apply this formalism to derive series expansions for interactions between flat and curved particle pairs, multibody interactions, as well as corrections to all these interactions due to thermal fluctuations.

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1. Introduction

In 1973 Wolfgang Helfrich proposed a continuum-elastic Hamiltonian that describes fluid lipid membranes as curvature-elastic surfaces [1]. This seminal work, and Helfrich's many subsequent ingenious contributions, opened up an incredibly fruitful field of research for many scientists, and it has been instrumental in disciplines ranging from soft matter physics over chemical engineering and materials science to biophysics and cell biology. In this review we will deal with one particular consequence of this Hamiltonian, namely the fact that surface-associated objects will generally experience interactions mediated by the membrane's curvature-elastic deformations and fluctuations thereof. While this phenomenon has been subject of numerous studies in the past, the aim of the present review is to illustrate how those interactions can be described systematically and at arbitrary accuracy within the physically intuitive framework of Effective Field Theory (EFT).

1.1. The Helfrich Hamiltonian

At its heart, the Helfrich Hamiltonian is a functional that assigns an energy to a two-dimensional surface embedded in three-dimensional space:

$$H[\mathcal{S}] = \int_{\mathcal{S}} \mathrm{d}A \left\{ \frac{1}{2}\kappa(K-K_0)^2 + \overline{\kappa} K_G \right\}. \tag{1}$$

Here, $K = c_1 + c_1$ is the total curvature and $K_G = c_1 c_2$ is the Gaussian curvature of a given point on the surface, at which c_1 and c_2 are the local principal curvatures. The inverse length K_0 is the bilayer's spontaneous curvature (which often vanishes due to up–down symmetry) and the two characteristic energies κ and $\overline{\kappa}$ are the mean and Gaussian curvature moduli, respectively. The simplicity of this functional reflects several properties of fluid membranes: They are close to inextensible while fairly easy to bend, so curvature deformations are the soft modes which an elastic description should capture. And due to fluidity shear stresses must vanish, requiring a functional that is insensitive to purely tangential deformations.

A functional variation of Eq. (1) leads to the associated Euler–Lagrange equation, whose solutions describe equilibrium (or "ground state") shapes of fluid membranes, subject perhaps to extra constraints [2]:

$$\kappa\left\{ -\Delta K + \frac{1}{2}K'\left[K'K - 2\left(K^2 - K_G\right)\right]\right\} + \sigma K = P\ , \tag{2}$$

where $K' = K - K_0$, the tension σ and the pressure P are Lagrange multipliers fixing area and volume constraints, respectively, and Δ is the covariant Laplacian on the surface. Uncovering the subtle physics hidden in this fourth order partial nonlinear differential equation has been a major research thrust for many years [3,4].

1.2. Geometric field theory

The Helfrich Hamiltonian (1) is a continuum-elastic energy functional. However, one can also interpret it as the action of a field theory, for which the geometry of the membrane is the field—much in the same sense in which general relativity is a field theory built on the geometry of space-time. With this viewpoint, Eq. (2) is the free field equation, and just like its general relativity counterpart (and for the same reason) it is highly nonlinear and very difficult to deal with. But general relativity has a weak field limit, Newtonian gravity, and the equivalent simplification for membranes occurs in the limit of weakly deformed surfaces. For instance, if a membrane does not deviate much from a flat plane, the Hamiltonian (1) strongly simplifies to

$$H[\mathcal{S}] = \int_{\mathcal{S}_p} \mathrm{d}^2x \left\{ \frac{1}{2}\kappa(\mathrm{Tr}\,\boldsymbol{h})^2 + \overline{\kappa}\det\boldsymbol{h} \right\}\ , \tag{3}$$

where $(\boldsymbol{h})_{ij} = \partial^2 h/\partial x^i \partial x^j$ is the Hessian of the height function $h(x,y)$, and the integral now extends over the projection $\mathcal{S}_p$ of the surface $\mathcal{S}$ onto the base plane. The parameterization via $h(x,y)$ is called Monge gauge [5], and Eq. (3) is then referred to as the Helfrich Hamiltonian in *linearized* Monge gauge. Linearized, because all terms in this gauge that would go beyond quadratic order have been neglected, such that the field equations are now linear:

$$-\kappa\nabla^2\nabla^2 h + \sigma\nabla^2 h = 0\ , \tag{4}$$

where ∇^2 is now the ordinary Laplacian on the flat base plane and we assumed no pressure difference between the two sides of the membrane (in accordance with the condition of approximate flatness).[1]

While the "true" field theory is the nonlinear version resting on Eqs. (1) and (2), the linearized version is often a good approximation. What might at first sound surprising, though, is that *even* the linear theory can be difficult to deal with. The problem of membrane mediated interactions is a prime example where this happens, and we thereby home in on the central topic of our review.

1.3. Curvature charges and their interactions

Field theories become more interesting once there are charges to which the field can couple. Since charges source fields, and fields in turn exert forces on charges, this coupling induces interactions between the charges, mediated by the field. For instance, electric charges interact

[1] If in this parameterization one includes the spontaneous curvature via $\frac{1}{2}\kappa\left(\nabla^2 h - K_0\right)^2$, it will only give rise to a constant $\frac{1}{2}\kappa K_0^2$ and the term $\kappa K_0 \nabla^2 h$, which can be integrated to the boundary; for the most common boundary conditions K_0 hence becomes irrelevant, even if the nonlinear theory has a non-vanishing K_0. However, a systematic expansion of Eq. (1) up to quadratic order in h shows that $\frac{1}{2}\kappa K_0^2$ in fact gets multiplied by $1 + \frac{1}{2}(\nabla h)^2$ (from the area element), resulting in the additional term $\frac{1}{4}\kappa K_0^2(\nabla h)^2$, which rescales the surface tension. The connection between spontaneous curvature and tension (also beyond Monge gauge) is further discussed in Ref. [6].