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Brief paper

An insight into instrumental variable frequency-domain subspace identification[☆]

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ABSTRACT

In this paper, we study instrumental variable subspace identification of multi-input/multi-output linear-time-invariant, discrete-time systems from non-uniformly spaced frequency response measurements. A particular algorithm, which does not require noise covariance function to be known *a priori* is shown to be strongly consistent provided that disturbances have uniformly bounded second-order moments and the frequencies satisfy a certain regularity condition. Interpolation properties of this algorithm and a related one are also studied. A numerical example illustrating the properties of the studied algorithms is presented.

1. Introduction

Let us consider a multi-input/multi-output, linear-time-invariant, discrete-time system represented by the state–space equations

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^m$, and $y(k) \in \mathbf{R}^p$ are respectively the state, the input, and the output of the system. Here, the sets of the real and the complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. The transfer function of the system in Eq. (1) denoted by $G(z)$ is calculated as

$$G(z) = D + C(zI_n - A)^{-1}B \tag{2}$$

where I_n is the $n \times n$ identity matrix. We assume that the system in Eq. (1) is stable and the pairs (A, B) and (C, A) are controllable and observable, respectively. The stability of the system in Eq. (1) means that $G(z)$ is a proper rational matrix that is analytic and bounded in the region $\{z \in \mathbf{C} : |z| \geq 1\}$ and both the controllability and the observability of the pairs (A, B) and (C, A) mean that (A, B, C, D) is a minimal realization of $G(z)$.

In this paper, we study the problem of identifying $G(z)$ from its N noise-corrupted samples on the unit circle:

$$G_k = G(e^{i\omega_k}) + \eta_k, \quad k = 1, \ldots, N. \tag{3}$$

We require algorithms recovering $G(z)$ be *strongly consistent* and *interpolatory*. Recall that an algorithm producing the true model from a finite amount of data when the noise is zero is called interpolatory. Strong consistency is a most natural requirement for any useful algorithm: as the amount of data increases, the estimate should improve and asymptotically the correct model should be obtained. In practice, any algorithm will have to use a finite amount of data. Interpolation property is particularly important for lightly damped systems (McKelvey, Akçay, & Ljung, 1996a,b).

In McKelvey et al. (1996a), two subspace identification algorithms, which are both strongly consistent and interpolatory, were developed. The first algorithm uses uniformly spaced frequency response data and the noise is required only to have bounded moments of order two. The second algorithm uses non-uniformly spaced frequency response data and strong consistency is achieved under more restrictive assumptions: the noise has bounded moments of order four and its covariance function be known *a priori*. The purpose of the paper is to relax the latter convergence requirements when the frequency response data are given on non-uniform grids of frequencies.

In McKelvey (1997), a frequency-domain subspace algorithm based on the instrumental variables technique (Söderström & Stoica, 1989) was developed. This algorithm is consistent without requiring the knowledge of the noise covariance matrix provided that a certain rank constraint is satisfied. A similar technique was used in the time-domain subspace algorithms in Verhaegen (1993, 1994). More recently in Pintelon (2002), asymptotic properties of

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the subspace identification algorithms in McKelvey et al. (1996a) and Van Overschee and De Moor (1996) were studied when the true noise covariance matrix is replaced by the sample noise covariance obtained from independent repeated experiments.

The current paper was motivated from time delay estimation problem. Time delay estimation between signals acquired by different sensors is intrinsic in many signal processing problems (Carter, 1983). In frequency-domain, single delay estimation amounts to recovering a scalar transfer function from phase measurements. When the corruptions in the phase data are independent, the latter problem can be viewed as estimating a rational function from independent frequency response measurements with the noise covariance scaled by the squared-magnitude of the unknown transfer function. Strong consistency will not be achieved by parametric identification techniques unless the noise and the system transfer functions are jointly parameterized. On the other hand, when the phase data are given on equidistant frequencies, strong consistency can be achieved by the subspace techniques. The details can be found in Akçay (in press).

This paper is organized as follows. In Section 2, first the basic equations used by the frequency-domain subspace algorithms are developed. Then, an instrumental variable subspace algorithm is proposed. A key role in designing an instrumental variable is played by the well-known QR-decomposition of an extended Vandermonde matrix constructed from complex exponentials on the unit circle. Next, the noise and the frequency assumptions are put forward and an instrumental variable identification algorithm is outlined. It is shown that the proposed algorithm is both strongly consistent and interpolatory if the frequencies are in one-to-one correspondence with the uniformly spaced frequencies on the unit circle. The interpolation property of this algorithm is not explicit. Then, the QR-decomposition is applied to a full-size Vandermonde matrix yielding explicit interpolation relation between the number of the data and the model order. In Section 3, the properties of the studied algorithms are illustrated by means of a simulation example. Section 4 concludes the paper.

1.1. Notation

Let $\Re(X)$ and $\Im(X)$, X^T, $\overline{X}$, and X^H denote respectively the real and the imaginary parts, the transpose, the complex conjugate, and the complex conjugate transpose of a given matrix X. The kth largest singular value of X is denoted by $\sigma_k(X)$. The Frobenius and the sup or max norms of X are defined respectively by $\|X\|_F = \sqrt{\sum_{k,l}|X_{k,l}|^2}$ and $\|X\|_\infty = \max_{k,l}|X_{k,l}|$. When X is of full-column rank, its Moore–Penrose pseudo-inverse is defined by $X^\dagger = (X^TX)^{-1}X^T$. The Kronecker product of X and Y are denoted by $X \otimes Y$. Adopting the MATLAB notation, $Y(r, :)$ and $Y(:, k)$ will denote respectively the rth (block) row and the kth (block) column of a given (block) matrix Y. The supremum norm of a complex transfer matrix with bounded elements on the unit circle is defined by $\|G\|_\infty = \sup_{\omega\in\mathbf{R}} \sigma_1 G(e^{i\omega})$. Given a set S, its cardinality is denoted by $\aleph(S)$. We set $z_k = e^{i\omega_k}$ for $k = 1, \ldots, N$ and, without loss of generality, assume that the frequencies satisfy $\omega_1 \le \omega_k \le \omega_N$. Let

$$M = 2\aleph\left(\{z_k : z_k \notin \mathbf{R}\}\right) + \aleph\left(\{z_k : z_k \in \mathbf{R}\}\right). \tag{4}$$

The notation $y = O(x)$ means that $|y/x|$ is asymptotically bounded. The expected value of a given random variable x is denoted by $\mathbf{E}[x]$. The Kronecker delta is denoted by δ_{ks}.

2. An instrumental variable subspace identification algorithm

By shifting $G(z)$ in Eq. (2) q samples forward, using Eq. (1) recursively with $U(z) = I_m$, and exploiting the fact that the system described by Eq. (1) has a real-valued impulse response, the following formula

$$\mathcal{G} = \mathcal{O}_q\mathcal{X} + \Gamma\mathcal{W}_q + \mathcal{N} \tag{5}$$

can be derived as in McKelvey et al. (1996a) where q is a fixed design variable to be specified later and

$$\mathcal{G} = \begin{bmatrix}\Re(\mathcal{G}_c) & \Im(\mathcal{G}_c)\end{bmatrix}, \tag{6}$$

$$\mathcal{X} = \begin{bmatrix}\Re(\mathcal{X}_c) & \Im(\mathcal{X}_c)\end{bmatrix}, \tag{7}$$

$$\mathcal{W}_q = \begin{bmatrix}\Re(\mathcal{V}_q) & \Im(\mathcal{V}_q)\end{bmatrix}, \tag{8}$$

$$\mathcal{N} = \begin{bmatrix}\Re(\mathcal{N}_c) & \Im(\mathcal{N}_c)\end{bmatrix}, \tag{9}$$

$$\mathcal{G}_c = \frac{1}{\sqrt{N}}\begin{bmatrix} G_1 & \cdots & G_N \\ \vdots & \ddots & \vdots \\ z_1^{q-1}G_1 & \cdots & z_N^{q-1}G_N \end{bmatrix}, \tag{10}$$

$$\mathcal{X}_c = \frac{1}{\sqrt{N}}\begin{bmatrix}X(z_1) & \cdots & X(z_N)\end{bmatrix}, \tag{11}$$

$$\mathcal{V}_q = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{q-1} & \cdots & z_N^{q-1} \end{bmatrix} \otimes I_m, \tag{12}$$

$$\mathcal{N}_c = \frac{1}{\sqrt{N}}\begin{bmatrix} \eta_1 & \cdots & \eta_N \\ \vdots & \ddots & \vdots \\ z_1^{q-1}\eta_1 & \cdots & z_N^{q-1}\eta_N \end{bmatrix}, \tag{13}$$

$$\mathcal{O}_q = \begin{bmatrix} C \\ \vdots \\ CA^{q-1} \end{bmatrix}, \tag{14}$$

$$\Gamma = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{q-2}B & CA^{q-3} & \cdots & D \end{bmatrix}, \tag{15}$$

$$X(z) = (zI_n - A)^{-1}B. \tag{16}$$

The term $\Gamma\mathcal{W}_q$ in Eq. (5) can be removed by the use of the orthogonal projection matrix onto the null-space of $\mathcal{W}_q$:

$$\Pi_{N,q}^{\perp} = I_{2mN} - \mathcal{W}_q^T(\mathcal{W}_q\mathcal{W}_q^T)^{-1}\mathcal{W}_q \tag{17}$$

assuming that the inverse exists. Thus,

$$\mathcal{G}\Pi_{N,q}^{\perp} = \mathcal{O}_q\mathcal{X}\Pi_{N,q}^{\perp} + \mathcal{N}\Pi_{N,q}^{\perp}. \tag{18}$$

The column size of $\mathcal{N}\Pi_{N,q}^{\perp}$ grows in proportion to N as N increases and makes Algorithm 2 in McKelvey et al. (1996a) biased unless the noise covariance information is utilized in the algorithm. In an effort to remove this bias, without access to the noise covariance information, it was suggested in McKelvey (1997) to use instrumental variables in the basic subspace algorithm. An instrumental variable denoted by $\mathcal{Q}_2$ should satisfy the following two properties:

$$\operatorname{rank}(\mathcal{X}\Pi_{N,q}^{\perp}\mathcal{Q}_2) = n, \quad \text{for all } N \ge N_0 \tag{19}$$

for some $N_0 > 0$ and as $N \to \infty$,

$$\|\mathcal{N}\Pi_{N,q}^{\perp}\mathcal{Q}_2\|_F \to 0 \quad \text{(w.p.1)}. \tag{20}$$