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Time-invariant uncertain systems: A necessary and sufficient condition for stability and instability via homogeneous parameter-dependent quadratic Lyapunov functions☆

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ABSTRACT

This paper investigates linear systems with polynomial dependence on time-invariant uncertainties constrained in the simplex via homogeneous parameter-dependent quadratic Lyapunov functions (HPD-QLFs). It is shown that a sufficient condition for establishing whether the system is either stable or unstable can be obtained by solving a generalized eigenvalue problem. Moreover, this condition is also necessary by using a sufficiently large degree of the HPD-QLF.

1. Introduction

Various methods have been proposed for the stability of linear systems with time-invariant uncertainty constrained in a polytope. Generally, these methods exploit parameter-dependent Lyapunov functions and LMIs; see e.g. Leite and Peres (2003), which considers Lyapunov functions with linear dependence, Bliman (2004), which proposes Lyapunov functions with polynomial dependence, Chesi, Garulli, Tesi, and Vicino (2005), which introduces the class of HPD-QLFs, Scherer (2006), which proposes a general framework for LMI relaxations, Oliveira and Peres (2007), where homogeneous solutions are characterized, Lavaei and Aghdam (2008), which addresses the case of semi-algebraic sets, and Oishi (2009) and Peaucelle and Sato (2009), where matrix-dilation approaches are considered.

Some of these methods provide necessary and sufficient conditions for robust stability. However, the necessity is achieved for an unknown degree of the polynomials used. This implies that, if the system is unstable, no conclusion can be reached. This paper addresses this problem via homogeneous parameter-dependent quadratic Lyapunov functions (HPD-QLFs) for the case of polynomial dependence on the uncertainty. It is shown that a sufficient condition for establishing either stability or instability can be obtained by solving a generalized eigenvalue problem, and that this condition is also necessary by using a sufficiently large degree of the HPD-QLF. The idea behind this condition is to exploit the LMI relaxation introduced in Chesi et al. (2005) via the square matrical representation (SMR)[1] in order to characterize the instability via the presence of suitable vectors in certain eigenspaces.

Before proceeding, it is worth explaining that the proposed approach differs from Chesi (2005), which proposes a non-Lyapunov method for establishing stability and instability of uncertain systems, from Chesi (2007), which investigates robust $\mathcal{H}_\infty$ performance via eigenvalue problems, from Ebihara, Onishi, and Hagiwara (2009), which exploits D/G-scaling in the case of rational dependence on the uncertainty, from Masubuchi and Scherer (2009), which derives a recursive algorithm based on the linear fractional representation, and from Goncalves, Palhares, Takahashi, and Mesquita (2006, 2007), which propose a branch-and-bound method in the case of linear dependence on the uncertainty.

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[1] The SMR allows one to establish if a polynomial is a sum of squares (SOS) of polynomials via an LMI; see e.g. Chesi (2007), Chesi, Garulli, Tesi, and Vicino (2003), Chesi, Garulli, Tesi, and Vicino (2009) and Chesi, Tesi, Vicino, and Genesio (1999) and references therein.

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2. Preliminaries

Notation. $\mathbb{R}, \mathbb{C}$: real and complex numbers; $\mathbb{R}_0$: $\mathbb{R} \setminus \{0\}$; I_n: $n \times n$ identity matrix; $A > 0$: symmetric positive definite matrix; $A \otimes B$: Kronecker's product; A', $\mathrm{tr}(A)$, $\det(A)$: transpose, trace and determinant of A; $\mathrm{vec}(A)$: vector with the columns of A stacked below each other; $\mathrm{spc}(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$; $\mathrm{span}(v_1, \ldots, v_k) = \{a_1 v_1 + \cdots + a_k v_k,\ a_1, \ldots, a_k \in \mathbb{R}\}$; $\mathrm{sq}(p) = \left(p_1^2, \ldots, p_q^2\right)'$ with $p \in \mathbb{R}^q$; CT, DT: continuous-time and discrete-time; s.t.: subject to.

Let us consider the uncertain system

$$\begin{cases} \text{(CT case)} & \dot{x}(t) = A(p)x(t) \\ \text{(DT case)} & x(t+1) = A(p)x(t) \end{cases} \quad \forall p \in \mathcal{P} \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $p \in \mathbb{R}^q$ is the uncertain parameter, and $\mathcal{P}$ is the simplex, i.e. $\mathcal{P} = \left\{p \in \mathbb{R}^q : \sum_{i=1}^q p_i = 1,\ p_i \geq 0\right\}$. The function $A : \mathbb{R}^q \to \mathbb{R}^{n\times n}$ is a *matrix form* of degree m_A, i.e. a matrix whose entries are forms (i.e. homogeneous polynomials) of degree m_A. Let us define

$$\mathcal{A} = \left\{A(p) \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\right\}. \tag{2}$$

In what follows we will say that:

- (CT case) $A(p)$ is stable if and only if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \mathrm{spc}(A(p))$;
- (DT case) $A(p)$ is stable if and only if $|\lambda| < 1$ for all $\lambda \in \mathrm{spc}(A(p))$;
- $\mathcal{A}$ is stable if and only if $A(p)$ is stable for all $p \in \mathcal{P}$;
- $\mathcal{A}$ (resp., $A(p)$) is unstable if $\mathcal{A}$ (resp., $A(p)$) is not stable according to the previous definitions.

In order to introduce the proposed result, let us define

$$\begin{array}{ll} \text{(CT case)} & d = m_A \\ \text{(DT case)} & d = 2m_A \end{array} \tag{3}$$

and let $p^{\{m\}}$ be a vector containing all monomials of degree m in p. Let us introduce

$$\begin{aligned} \mathcal{S} &= \Big\{S = S' : \Delta\left(p^{\{m\}}, S\right) \text{ does not contain} \\ &\qquad \text{monomials } p_1^{i_1} \ldots p_q^{i_q} \text{ with at least one } i_j \text{ odd}\Big\} \\ \mathcal{U} &= \left\{U = U' : \Delta\left(p^{\{m+d\}}, U\right) = 0\ \forall p\right\} \end{aligned} \tag{4}$$

where the notation $\Delta(\cdot,\cdot)$ means

$$\Delta(b, B) = (b \otimes I_n)' B (b \otimes I_n) \tag{5}$$

for any suitable b, B, and let $S(\beta)$, $U(\alpha)$ be linear parameterizations of $\mathcal{S}$, $\mathcal{U}$. Define the functions

$$P(\mathrm{sq}(p), \beta) = \Delta\left(p^{\{m\}}, S(\beta)\right) \tag{6}$$

and

$$\begin{array}{ll} \text{(CT case)} & Q(\mathrm{sq}(p), \beta) = -B'C - CB \\ \text{(DT case)} & Q(\mathrm{sq}(p), \beta) = \left(\sum_{i=1}^q p_i^2\right)^d C - B'CB \end{array} \tag{7}$$

for $B = A(\mathrm{sq}(p))$ and $C = P(\mathrm{sq}(p), \beta)$. Let $R(\beta)$ be an SMR matrix of the matrix form $Q(\mathrm{sq}(p), \beta)$, i.e. a symmetric function satisfying

$$\Delta\left(p^{\{m+d\}}, R(\beta)\right) = Q(\mathrm{sq}(p), \beta). \tag{8}$$

The following theorem is given in Chesi (2008), Chesi et al. (2005) and Chesi et al. (2009) and investigates the robust stability of (1) via an HPD-QLF of degree m, i.e. via a Lyapunov function of the form $x'P(p, \beta)x$.

Theorem 1 (*Chesi, 2008*). *The set $\mathcal{A}$ is stable if and only if there exists m such that the following LMIs hold for some α, β:*

$$\begin{cases} S(\beta) > 0 \\ R(\beta) + U(\alpha) > 0. \end{cases} \tag{9}$$

3. Stability and instability condition

Let us define

$$T(\beta) = \Delta\left(K, I_{dq} \otimes S(\beta)\right) \tag{10}$$

where K is the matrix satisfying

$$\underbrace{p \otimes \cdots \otimes p}_{d \text{ times}} \otimes p^{\{m\}} = K p^{\{m+d\}} \tag{11}$$

(see also (18) for a key property of $T(\beta)$) and define

$$\eta^* = \sup_{\alpha,\beta,\eta} \eta \quad \text{s.t.} \begin{cases} S(\beta) > 0 \\ R(\beta) + U(\alpha) - \eta T(\beta) > 0 \\ \mathrm{tr}(S(\beta)) = 1. \end{cases} \tag{12}$$

Let us define

$$V = R(\beta^*) + U(\alpha^*) \tag{13}$$

where α^*, β^* are optimal values of α, β in (12), and let $c_1, \ldots, c_r$ be the eigenvectors of the non-positive eigenvalues of V, i.e.

$$\begin{cases} c_i' c_i = 1 \\ V c_i = \lambda_i c_i \quad \text{for some } \lambda_i \in \mathbb{R}, \lambda_i \leq 0. \end{cases} \tag{14}$$

Theorem 2. *The set $\mathcal{A}$ is stable if and only if there exists m such that $\eta^* > 0$. Moreover, $\mathcal{A}$ is unstable if and only if there exist m and $(u, y) \in \mathbb{R}_0^q \times \mathbb{R}_0^n$ such that $A(\xi(u))$ is unstable and*

$$u^{\{m+d\}} \otimes y \in \mathrm{span}\{c_1, \ldots, c_r\} \tag{15}$$

where $\xi : \mathbb{R}_0^q \to \mathcal{P}$ is the function

$$\xi(u) = \left(\sum_{i=1}^q u_i^2\right)^{-1} \mathrm{sq}(u). \tag{16}$$

Proof. Let us consider the stability statement, and let us observe that K in (11) is a full column rank (see e.g. Chesi et al. (2005)), which directly implies from (10) that

$$T(\beta) > 0 \iff S(\beta) > 0. \tag{17}$$

Therefore, the stability statement follows from (17) and Theorem 1. Indeed, observe that the constraint $\mathrm{tr}(S(\beta)) = 1$ is not restrictive since $S(\beta)$, $R(\beta)$, $U(\alpha)$ and $T(\beta)$ are linear functions, and it is introduced in order to normalize the solution of (12).

Let us consider the instability statement. The sufficiency is obvious because, if $A(\xi(u))$ is unstable and $\xi(u) \in \mathcal{P}$, then $\mathcal{A}$ is unstable for definition. Hence, let us consider the necessity and let us assume that $\mathcal{A}$ is unstable. From the stability statement, we have $\eta^* \leq 0$. Observe that

$$\Delta\left(p^{\{m+d\}}, T(\beta)\right) = \left(\sum_{i=1}^q p_i^2\right)^d P(\mathrm{sq}(p), \beta) \tag{18}$$

and let us suppose for contradiction that, for all m, (15) does not hold.

Let us consider firstly the CT case. This supposition implies that $\mathrm{Re}(\lambda) < -0.5\eta^*$ for all $\lambda \in \mathrm{spc}(A(p))$ for all $p \in \mathcal{P}$. In fact, from (6)–(8) and (18) and Lemma 3 in Chesi et al. (2005), the first two constraints in (12) imply that

$$\begin{cases} P(p) > 0 \\ Q(p) - \eta P(p) > 0 \end{cases} \quad \forall p \in \mathcal{P}. \tag{19}$$

Consequently, there exists $\varepsilon > 0$ such that $A(p) + 0.5(\eta^* + \varepsilon)I$ is stable for all $p \in \mathcal{P}$. Let us replace $A(p)$ with $A(p) + 0.5(\eta^* + \varepsilon)I$ in our original problem. It follows that the new set $\mathcal{A}$ is stable, and the